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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ON AN INEQUALITY OF PÓLYA submitted by BRIAN DOHERTY in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis concerns the linear n^{th} order differential operator with leading coefficient 1 and the disconjugacy of the operator on certain intervals. Of particular interest is the case of intervals with singular endpoints where the usual notion of a zero of a function breaks down and with it the meaning of disconjugacy on such an interval. In this regard the concept of a generalized zero as developed by Levin is discussed and the development of generalized derivatives is given from Willett and Muldowney. The application of this to the definition of disconjugacy on the closed interval in case of singular endpoints and a necessary and sufficient condition in terms of Descartes systems of solutions for disconjugacy is given.

A version of a theorem of Muldowney may be stated as: L is an n^{th} order linear differential equation with leading coefficient 1 which is disconjugate on $[a,b]$, f is an n times differentiable function on (a,b) with $f^{(j)}(t_i) = 0$, $j = 0,1,\ldots,m_i-1$, $i = 1,2,\ldots,\ell$ where $t_1,\ldots,t_\ell$, are ℓ points in $[a,b]$ and $m_1,\ldots,m$ are positive integers such that $m_1 + m_2 + \ldots + m = k \leq n$; L_{n-k} is an $n-k^{th}$ order operator such that a basis for the solution set of $L_{n-k}\, x = 0$ on (a,b) is $\{x : Lx = 0,\ x^{(j)}(t_i) = 0,\ j = 0,1,\ldots,m_i-1,\ i = 1,2,\ldots,\ell\}$, $pk = \prod_{i=1}^{\ell} (t-t_i)^{m_i}$. If $Lf > 0$ on $[a,b)$ then $pk\, L_{n-k} f > 0$ also. This is a generalization of a theorem of Pólya. Both these theorems are given and proved by an alternate method. The method was suggested

by Pólya.

Finally some corollaries of the Muldowney theorem are used to obtain bounds on general first, second and third order operators in terms of special first, second and third order linear operators with constant coefficients.

The thesis is largely expoitory. However, the proof of the generalized Pólya theorem is new. Several of the applications in Chapter IV are also new.

TABLE OF CONTENTS

CHAPTER I

ABSTRACT

The first chapter has two main objectives.

The first objective is to overcome the problem of defining disconjugacy in cases where one or both of the endpoints of the interval in question are singular. This problem is resolved in the work of Levin and this first chapter is an exposition on his work. Levin conceived the concept of a generalized zero using principal systems of solutions. These generalized zeros allow for the extension of the definition of disconjugacy to intervals with singular endpoints.

The second objective is to show that every n^{th} order linear differential operator with leading coefficient 1 which is disconjugate on an interval $[a,b]$ has a fundamental principal system of solutions which is Descartes on (a,b) and that the existence of such a system of solutions for such an operator implies the disconjugacy of that operator on $[a,b]$.

CHAPTER I

INTRODUCTION

Linear Differential Operators

Let I be an interval of the real line. The general n^{th} order linear differential operator L is defined by $Lx(t) = x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \ldots + p_n(t)x(t)$. The functions $p_i(t)$ are continuous real valued functions with domain I. A solution of the equation $Lx = 0$ is a function $x(t)$ which is n times continuously differentiable and which satisfies the equation $Lx(t) = 0$ for all t in I.

Alternatively, L is defined with the $p_i(t)$ being elements of $\mathrm{loc}\, L^1(I)$ in which case $x(t)$ is a solution of $Lx = 0$ if x is in $\mathrm{loc}\, AC^{n-1}(I)$ and $Lx(t) = 0$ almost everywhere in I. The two definitions are not the same and here the first is used except where it is otherwise stated.

A function $f(t)$ is said to have a zero of multiplicity k at a point t_o if: $f(t_o) = f'(t_o) = \ldots = f^{(k-1)}(t_o) = 0$ and $f^{(k)}(t_o) \neq 0$ or $f^{(k)}(t_o)$ does not exist. The symbolism $Z_f(t_o) = k$ denotes this fact. The total number of zeros of f on the interval I is $Z_f(I) = \sum_{t \in I} Z_f(t)$. The concept of the number of zeros of a function gives a means of classifying n^{th} order linear differential operators.

If all nontrivial solutions of $Lx = 0$ are such that $Z_x(I) \leq n - 1$ then L is called *disconjugate* on the interval I. The interval is important here. For example $Lx = x'' + x$ is disconjugate on $[0,1]$ but not on $[0,\pi]$. In fact Proposition 1 of Chapter 3 of [1] proves

that given a compact interval I and a linear differential operator L, there is a number $\delta > 0$ so that L is disconjugate on every subinterval of I of length less than δ.

If the interval I is open then two kinds of endpoints are distinguished with respect to an operator L. If the endpoint is $\pm\infty$ or if on some neighborhood of the endpoints relative to $\bar{I}$, the closure of I; at least one of the $p_i(t)$ is not integrable then the endpoint is called singular. In all other cases the endpoint is nonsingular. Endpoints are also classified in another way. An endpoint of I is nonoscillatory if the number of zeros of every nontrivial solution of $Lx = 0$ is less than n in some neighborhood, relative to $\bar{I}$, of the endpoint. In this sense $Lx = x'' + x$ does not have a nonoscillatory endpoint at ∞ since all solutions of $Lx = 0$ are of the form $a \sin t + b \cos t$ where a, b are constant. But $a \sin t + b \cos t$ is zero when $\tan t = -b/a$ which happens once every half open interval of length π.

The two classifications of endpoints are not completely unrelated for all non-singular endpoints are nonoscillatory.

Principal Systems of Solutions and Generalized Zeros

The equation $Lx = 0$ has a system of n linearly independent solutions for L of order n. On an interval I it may be possible to find a set of solutions $\{u_1, u_2, \ldots, u_n\}$ which are not only linearly independent solutions of $Lx = 0$ but which are strictly positive on a neighborhood of one of the endpoints and for which the limit $u_k(t)/u_{k+1}(t)$; $k = 1,2,\ldots,n-1$ as t approaches that endpoint is zero. Such a system of solutions is called a principal system of solutions at the endpoint.

The next lemmas discuss this possibility.

LEMMA 1.1 *Let* L *be an* n^{th} *order linear differential operator with coefficients in* $C(a,b)$ *and let* b *be a nonoscillatory endpoint for* L. *If* $z_1(t)$ *and* $z_2(t)$ *are linearly independent solutions of* $Lx = 0$ *which are strictly positive on a neighborhood* (c,b) *of* b, *then* $\lim_{t \to b^-} f(t)$ *exists and is in* $[0,\infty]$ *where* $f(t) = z_1(t)/z_2(t)$.

Proof. Since z_1 and z_2 are positive on (c,b) $0 \leq \liminf_{t \to b^-} f(t) \leq \limsup_{t \to b^-} f(t) \leq \infty$. If $\liminf_{t \to b^-} f(t) < d < \limsup_{t \to b^-} f(t)$ then $f(t) = d$ for some t in every neighborhood of b. Therefore $z(t) = z_1(t) - dz_2(t)$ which is a solution of $Lx = 0$ has a zero in every neighborhood of b. However b is assumed to be a nonoscillatory endpoint and so this cannot happen. This contradiction proves the lemma.

The next lemma is Lemma 2.1 of Levin's paper [4]. This lemma provides for the existence of principal systems of solutions.

LEMMA 1.2 *If* c *is a nonoscillatory endpoint for* L *then* L *has a principal system of solutions at* c.

Proof. Let $I = (a,b)$ and let $c = b$. Let $\{z_1(t), z_2(t), \ldots, z_n(t)\}$ be a linearly independent set of solutions of $Lx = 0$. Since the endpoint b is assumed nonoscillatory each of the z_i has at most $n-1$ zeros on some neighborhood of b. Since there are at most a finite number of zeros for the z_i on that interval there is a point d in (a,b) such that none of the z_i have any zeros in (d,b). Therefore

each of the z_i is of constant sign on (d,b). Multiplying one of the z_i by -1 in the set $z_1, z_2, \ldots, z_n$ does not affect the linear independence of the set and so it may be assumed that all of the z_i are chosen to be positive on (d,b).

The proof now proceeds by induction. If $n = 2$, then let $f(t) = z_1(t)/z_2(t)$. The previous lemma shows the existence of a number $p \in [0,\infty]$ so that $\lim_{t \to b^-} f(t) = p$. If $p = 0$ then $\{z_1, z_2\}$ is a principal system at b. If $p = \infty$ then renumber so that z_1 becomes z_2 and z_2 becomes z_1 and then $\{z_1, z_2\}$ will be a principal system at b. When $p \neq 0$ and $p \neq \infty$ let $\bar{z}_2 = \pm (z_1 - pz_2)$ with $\pm$ choosen so that $\bar{z}_2 > 0$ on $(\bar{d}, b)$, where $\bar{d}$ is a point in (d,b). This can be done since b is a nonoscillatory endpoint and $\bar{z}_2$ is a solution of $Lx = 0$. But now $\{z_1, \bar{z}_2\}$ is a principal system at b.

Now suppose that $\{z_1, z_2, \ldots, z_k\}$ with $k < n$ satisfy the conditions of a principal system of solutions at b. Compare z_{k+1} with z_i, $i = 1,2,\ldots,k$; if $\lim_{t \to b^-} (z_{k+1}/z_i) = \infty$ for all i or $\lim_{t \to b^-} (z_{k+1}/z_i) = 0$ for all i add z_{k+1} to the list, in the first case at the end and in the second at the front. If $\lim_{t \to b^-} (z_{k+1}/z_i) = \infty$ for $i = 1,2,\ldots,j$ and $\lim_{t \to b^-} (z_{k+1}/z_i) = 0$ for $i = j+1, j+2, \ldots, k$ then put z_{k+1} into the list between z_j and z_{j+1}. When, however, for some i, $\lim_{t \to b^-} (z_{k+1}/z_j) = 0$, $j = i+1, i+2, \ldots, k$ and $\lim_{t \to b^-} (z_{k+1}/z_i) = q_i \in (0,\infty)$ put $\bar{z}_{k+1} = \pm(z_{k+1} - q_i z_i)$ with $\pm$ choosen to make $\bar{z}_{k+1} > 0$ on a neighborhood of b. Now redo the comparisons using $\bar{z}_{k+1}$ instead of z_{k+1}. Since $\lim_{t \to b^-} (\bar{z}_{k+1}/z_j) = 0$ for $j = i, i+1, \ldots, k$ the smallest index j so that $\lim_{t \to b^-} (\bar{z}_{k+1}/z_j) \in (0,\infty)$ is now at least $i-1$ and

so in a finite number of steps a set $\{z_1, z_2, \ldots, z_k, z_{k+1}\}$ results which is a principal system at b. This completes the induction. The observation that for $c = a$ the same technique will work finishes the proof.

The principal system $\{z_1, z_2, \ldots, z_n\}$ is not unique. For $c_{ii} > 0$ and y_i defined by $y_i = \sum_{j=1}^{i} c_{ij} z_j$, $\{y_1, y_2, \ldots, y_n\}$ is another principal system. The y_i are positive on a neighborhood of b because if $\left|\sum_{j=1}^{i-1} c_{ij} z_j\right| > c_{ii} z_i$ for some t in every neighborhood of b then $\lim_{t \to b^-} \left|\sum_{j=1}^{i-1} c_{ij} z_j\right| / (c_{ii} z_i) \geq 1$ but since $\lim_{t \to b^-} (z_j/z_i) = 0$ for $j < i$ that cannot be. The y_i satisfy the limit conditions since

$$\lim_{t \to b^-} \frac{\sum_{j=1}^{k} c_{kj} z_j}{\sum_{j=1}^{k+1} c_{k+1,j} z_j} = \lim_{t \to b^-} \frac{\sum_{j=1}^{k} c_{kj} (z_j/z_{k+1})}{\sum_{j=1}^{k+1} c_{k+1,j} (z_j/z_{k+1})} = 0 .$$

At nonsingular endpoints, principal systems of solutions may be obtained by imposing the initial conditions $z_i^{(j)}(b) = 0 \quad \forall j < n-i$ and $(-1)^{n-i} z_i^{(n-i)}(b) > 0$. If the principal system is defined this way then the solution z_i has $n-i$ zeros at b. It is in consideration of this fact that Levin introduces the concept of generalized zeros in [4].

Let b be a nonoscillatory endpoint for a linear differential operator L and let $\{z_1, z_2, \ldots, z_n\}$ be a principal system of solutions of $Lx = 0$ at b. A function $f(t)$ is said to have k generalized zeros at b when $k = \max\{j \mid \lim (f(t)/z_{n-j+1}) = 0,\ j = 1, 2, \ldots, n\}$.

LEMMA 1.3 *The concept of generalized zeros is well defined.*

Proof. Let $y_1, y_2, \ldots, y_n$ and $z_1, z_2, \ldots, z_n$ be two principal systems of solutions of $Lx = 0$ at b. Then $\lim_{t \to b^-} (y_1/z_1) = d_1 \in [0,\infty]$ by Lemma 1.1. If $d_1 = 0$ or ∞ then $Lx = 0$ has n+1 linearly independent solutions on a neighborhood of b and therefore $d_1 \in (0,\infty)$. Suppose for $i < k$ it is true that $\lim_{t \to b^-} (y_i/z_i) = d_i \in (0,\infty)$ then if $\lim_{t \to b^-} (y_k/z_k) = 0$ $\{z_1, z_2, \ldots, z_{k-1}, y_k, z_k, \ldots, z_n\}$ is a principal system of solutions of $Lx = 0$ at b. But for L of order n this is not possible and so $d_k \neq 0$. The same argument applied to $\lim_{t \to b^-} (z_k/y_k) = (1/d_k)$ implies $(1/d_k) \neq 0$ and so $d_k \in (0,\infty)$, $k = 1,2.,,,,n$.

Now if $f(t)$ is such that $k = \max\{j \mid \lim_{t \to b^-} (f/z_{n-j+1}) = 0;$ $j = 1,2,\ldots,n\}$ then since $\lim_{t \to b^-} (f/y_{n-j+1})$ exists and is

$$\lim_{t \to b^-} (f/z_{n-j+1}) \lim_{t \to b^-} (z_{n-j+1}/y_{n-j+1}) = d_{n-j+1} \lim_{t \to b^-} (f(t)/z_{n-j+1})$$

whenever $\lim_{t \to b^-} (f/z_{n-j+1})$ exists and since the same is true with z and y interchanged, k does not depend on the particular choice of principal system of solutions.

The proof for a left endpoint a is the same except that the limits are right hand limits. This completes the proof.

At nonsingular endpoints the concept of generalized zero corresponds to the ordinary concept of a zero of a function.

In the context of a given operator L the number of zeros $Z_f(b)$ at a singular endpoint b will be the number of generalized zeros of f at b. Therefore the number of zeros a function has at a particular

point will depend on a given linear differential operator. For example if $f(t) = e^{2t}$ then f has two zeros at ∞ with respect to $L = (D-3)(D-4)$ since $z_1 = e^{3t}$, $z_2 = e^{4t}$ is a principal system of solutions of $Lx = 0$ at ∞ and $\max\{j \mid \lim_{t\to\infty} (f/z_{n-j+1}) = 0,\ j = 1,2\} = 2$ since $\lim_{t\to\infty} (f/z_2) = 0$ and $\lim_{t\to\infty} (f/z_1) = 0$. However, if $L = (D-1)(D-3)$ f has only one zero at ∞ with respect to L since $z_1 = e^{t}$, $z_2 = e^{3t}$ is a principal system of solutions of $Lx = 0$ at ∞ and so $\max\{j \mid \lim_{t\to\infty} (f/z_{n-j+1}) = 0,\ j = 1,2\} = 1$ since only $\lim_{t\to\infty} (f/z_2)$ is zero; and if $L = (D-1)(D-2)$ then with respect to L, f has no zeros at ∞ for $z_1 = e^{t}$, $z_2 = e^{2t}$ is a principal system of solutions at ∞ for $Lx = 0$ and $\{j \mid \lim_{t\to\infty} (f/z_{n-j+1}) = 0,\ j = 1,2\}$ is the empty set since both $\lim_{t\to\infty} (f/z_1)$ and $\lim_{t\to\infty} (f/z_2)$ are nonzero. This is always the case, if $\{j \mid \lim_{t\to b^-} (f/z_{n-j+1}) = 0\}$ is an empty set f has no zeros at b.

Extension of the Definition of Disconjugacy

Once the notion of a zero of a solution is extended to the singular endpoints of an interval it becomes possible to extend the concept of disconjugacy to the closed interval with singular endpoints. In view of this an operator $Lx = x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x$ with $p_i(t) \in C(a,b)$ is disconjugate on $[a,b]$ if it is disconjugate on (a,b) and for every solution of the equation $Lx = 0$ it is true that $Z_x(a) + Z_x(b) + Z_x I \leq n-1$ where $I = (a,b)$. A similar definition applies to the half open intervals $(a,b]$ and $[a,b)$.

The next two lemmas relate disconjugacy on (a,b) to disconjugacy on $[a,b)$ and $(a,b]$. The Lemma 1.4 appears in Levin [4] as Lemma 2.2 where it is given without proof. The present proof is due to J.S. Muldowney.

<u>LEMMA 1.4</u> *Let* $I = (a,b)$. *If* $x(t) \in C^m I$ *and* $y(t) \in C^{m-1} I$ *and* $s \in I$ *with* $x(s) = x'(s) = \dots = x^{(m-1)}(s) = 0$ *and* $x^{(m)}(s) \neq 0$ *and* $y(s) = y'(s) = \dots = y^{(m-2)}(s) = 0$ *then there exists an* $\varepsilon > 0$ *such that if* $|c| < \varepsilon$ *then* $(x + cy)(t)$ *has at least* m *zeros in* I.

<u>Proof</u>. If $f(t)$ is a function such that $f(s) = 0$ and $f'(s) \neq 0$ then f must change sign at s since $f(t) = f'(s)(t-s) + o(t-s)$. Define the family of functions $\{f_c(t)\}_{c \in R}$ by

$$f_c(t) = [x(t) + cy(t)]/(t-s)^{m-1} \quad t \neq s$$

$$f_c(s) = cy^{(m-1)}(s)/(m-1)!$$

L'Hospital's rule implies $\lim_{t \to s} f_c(t) = cy^{(m-1)}(s)/(m-1)!$ and so $f_c(t)$ is continuous on I for each c. If $c = 0$, $f_o(s) = 0$ and $\lim_{t \to s} f_o'(t) = (x^{(m)}(s)/m!)$, again by L'Hospital's rule, so that $f_o(t)$ changes sign at s. $\{f_c(t)\}$ converges unifomily to $f_o(t)$ as $c \to 0$ on every compact subinterval of I and so $f_c(t)$ changes sign in I if c is sufficiently small. Therefore either $f_c(t_o) = 0$ for some $t_o \neq s$, $t_o \in I$ or $f_c(s) = 0 = cy^{(m-1)}(s)/(m-1)!$ and, in either case, since $x + cy$ has $m-1$ zeros at s by hypothesis, $x + cy$ has m zeros on I for small enough c and the lemma is proved.

The statement and proof of the next lemma are found in Levin [4] as Lemma 2.3.

LEMMA 1.5 *The linear differential operator* L *is disconjugate on* $(a,b]$ *and* $[a,b)$ *whenever it is disconjugate on* (a,b).

Proof. The proof is by induction. Define spaces M_k to be subsets of the solution space of $Lx = 0$ such that $x(t)$ is in M_k if $Z_x(b) \geq n-k$ and $Z_x(I) \geq k$ where $I = (a,b)$ and x is not identically zero on I. Having done this, the hypothesis of disconjugacy on I implies $M_n = \emptyset$. Suppose therefore that $M_n = M_{n-1} = \ldots = M_{k+1} = \emptyset$ and $M_k \neq \emptyset$. Let $x(t) \in M_k$; then $Z_x(b) \geq n-k$ and $Z_x(I) \geq k$, by the definition of M_k. The last inequality must be equality since if $Z_x(I) \geq k+1$ the fact that $Z_x(b) \geq n-k > n-(k+1)$ implies that $x(t) \in M_{k+1}$ which is assumed to be empty. Therefore $Z_x(I) = k$.

Let $t_i \in I$, $i = 1,2,\ldots,r$ be the points where $x(t)$ is zero. Let k_i be the multiplicities of these zeros. Therefore $Z_x(t_i) = k_i$ and $\sum_{i=1}^{r} k_i = k$. The solution $x(t)$ may be choosen so that r is the maximal number of distinct zeros in I for functions in M_k.

The hypothesis that L is disconjugate on I, implying as it does that b is a nonoscillatory endpoint, guarantees by Lemma 1.2 a principal system of solutions $\{u_1,u_2,\ldots,u_n\}$ for L at b. If $r < k$ construct a function $y(t) = \sum_{j=1}^{k+1} a_j u_j(t)$ from the set $\{u_1,u_2,\ldots,u_{k+1}\}$ so that $Z_y(t_i) \geq k_i-1$ and so that there are r numbers s_i in I different from the t_i such that $y(s_i) = 0$, $i = 1,2,\ldots,r$. This can be done since solving for a_j is solving k homogeneous equations in

$k+1$ unknowns. Having the solution $y(t)$ it can be noted that $y = \sum_{j=1}^{k+1} a_j u_j$ implies $Z_y(b) \geq n-k-1$. Further, $Z_y(b) = n-k-1$, since by the construction of y $Z_y I \geq k$ and so if $Z_y(b) \geq n-k$, then $y \in M_k$. But this cannot be since y has at least $r+1$ distinct zeros in I, because at least one $k_i > 1$ whenever $r < k$. This however is impossible since $x(t)$ with r distinct zeros in I had the maximal number in I. If $r = k$ take $y = u_{k+1}$. Here also $Z_y(b) = n-k-1$.

Next consider intervals (c_i,d_i) $i = 1,2,\ldots,r+1$ so that $(c_i,d_i) \cap (c_j,d_j) = \emptyset$ for $i \neq j$ and so that $t_i \in (c_i,d_i) \subset I$, $i = 1,2,\ldots,r$ and $b \in (c_{r+1},d_{r+1})$. Since b is nonoscillatory choose c_{r+1} so that $y(t)$ has no zeros in (c_{r+1},b). Form the function $v(t) = x(t)-qy(t)$ where q is a real constant. Lemma 1.4 implies that there exists an $\varepsilon > 0$ such that if $|q| < \varepsilon$ then $v(t)$ has at least k_i zeros on (c_i,d_i) $i = 1,2,\ldots,r$. Consider the function $x(t)/y(t)$ in (c_{r+1},b). Since $Z_x(b) \geq n-k > Z_y(b) = n-k-1$ the definition of generalized zero implies that $\lim_{t \to b} (x(t)/y(t)) = 0$. Therefore choose $s \in (c_{r+1},b)$ so that $|x(s)/y(s)| < \varepsilon$. Let $q = x(s)/y(s)$. Now the function $v(t) = x(t) - (x(s)/y(s))\, y(t)$ has k_i zeros in (c_i,d_i) $i = 1,2,\ldots,r$ and at least one zero in (c_{r+1},b) namely s. Therefore $Z_v(I) \geq k+1$ and $Z_v(b) \geq n-k-1$ implies that $v(t) \in M_{k+1}$. The facts that $x(t) = \sum_{j=1}^{k} b_j u_j$ and $y(t) = \sum_{j=1}^{k+1} a_j u_j$ with $a_{k+1} \neq 0$ which guarantees that $v(t)$ is not identically zero gives a contradiction and so the space M_k must be empty. The procedure for $[a,b)$ is the same, with the principal system being taken at a instead of at b. This completes the proof because all the M_k,

$k = 0,1,2,\ldots,n$ are empty and since any solution of $Lx = 0$ having n zeros on $(a,b]$ would be in one of the M_k no such solution not identically zero exists.

Markov and Descartes Systems

The principal system of solutions is used in connection with two other kinds of systems of functions which have certain properties described in terms of Wronskians. The Wronskian of functions $y_1,\ldots,y_k$ with $y_i \in C^{(k-1)}(I)$ is $W_k(y_1,y_2,\ldots,y_k) = \det [y_i^{(j-1)}]$ $i,j = 1,\ldots,k$. A Markov system of functions on an interval I is a set of functions $\{y_1, y_2, \ldots, y_m\}$ with $y_i \in C^{(m-1)}(I)$ satisfying $W_k(y_1, y_2, \ldots, y_k) > 0$ on I for $k = 1, 2, \ldots, m$. A Markov system with the further property that $W_k(y_{i_1},y_{i_2},\ldots,y_{i_k}) > 0$ for any subset of indices $\{i_j\}$ such that $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ is called a Descartes system. Levin in [4] calls such a system a + Descartes system.

In order to discuss Markov and Descartes systems it is useful to have certain facts about Wronskians. The next group of lemmas is from Coppel [1].

LEMMA 1.6 *If* $r(t) \neq 0$, *then* $W_k(ry_1,ry_2,\ldots,ry_k) = r^k W_k(y_1,y_2,\ldots,y_k)$ *This is Lemma 3 in Chapter III of* [1].

Proof. The quantity $(ry_j)^{(i)}$ is $ry_j^{(i)} + R_i$ where the R_i is a linear combination of $y_j,y_j',\ldots,y_j^{(i-1)}$ and coefficients depending on $r(t)$ independent of the y_j. The properties of determinants which allow the

adding of multiples of one column to another and the factoring out of elements common to a row or column therefore gives the lemma.

<u>COROLLARY 1.7</u> *If* $y_1(t) \neq 0$ *then* $W_k(y_1,y_2,\ldots,y_k) = y_1^k(t) \times W_{k-1}(z_1,z_2,\ldots,z_{k-1})$ *where* $z_i = (y_{i+1}/y_1)'$, $i = 1,2,\ldots,k-1$.
This is part of Lemma 3, Chapter III in Coppel [1].

<u>Proof</u>. Let $r(t)$ in the lemma be $y_1(t)$, then $W_k(y_1,y_2,\ldots,y_k)$ $= y_1^k(t)\, W_k(1, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \ldots, \frac{y_k}{y_1})$. Now expand the last Wronskian along the first column and obtain the corollary.

The next lemma is Lemma 4 of Chapter III of [1].

<u>LEMMA 1.8</u> *If* $W_{k-1}(y_1,y_2,\ldots,y_{k-1}) \neq 0$ *and* $W_k(y_1,y_2,\ldots,y_k) \neq 0$ *on* I *then*

$$\left[\frac{W_k(y_1,y_2,\ldots,y_{k-1},y)}{W_k(y_1,y_2,\ldots,y_k)}\right]' = \frac{W_{k-1}(y_1,y_2,\ldots,y_{k-1})\, W_{k+1}(y_1,y_2,\ldots,y_k,y)}{(W_k(y_1,y_2,\ldots,y_k))^2} .$$

<u>Proof</u>. Both sides of the desired equality are linear differential operators of order k . The solution of both equations formed by setting the operators equal to zero is the set $\{y_1,y_2,\ldots,y_k\}$ of linearly independent functions. Also the coefficient of $y^{(k)}$ is $W_{k-1}(y_1, y_2, \ldots,y_{k-1})/W_k(y_1, y_2, \ldots, y_k)$ in both cases. Therefore the two operators are the same operator and the lemma is proved.

The next lemma is Lemma 5 of Chapter III of [1].

LEMMA 1.9 *If* $W_{k-1}(y_1,y_2,\ldots,y_{k-1}) \neq 0$ *and* $W_{k-1}(y_2,y_3,\ldots,y_k) \neq 0$ *and* $W_k(y_1,y_2,\ldots,y_k) \neq 0$ *then*

$$W_{k-1}(y_2,\ldots,y_k)W_k(y_1,\ldots,y_{k-1},y) = W_{k-1}(y_1,\ldots,y_{k-1})W_k(y_2,\ldots,y_k,y) + W_{k-1}(y_2,\ldots,y_{k-1},y)W_k(y_1,\ldots,y_k) \quad .$$

Proof. The proof is essentially the same as the proof of the last lemma.

W.A. Coppel uses the last lemma and a nice induction to find a simpler criteria for a system to be Descartes. This is Proposition 4 of Chapter III of [1].

LEMMA 1.10 *If* $W_{j-i+1}(y_i,y_{i+1},\ldots,y_j) > 0$ *for all* $1 \leq i \leq j \leq n$ *then* $\{y_1,y_2,\ldots,y_n\}$ *is a Descartes system.*

Proof. The lemma is true for $n = 1$. For n larger than 1 assume that the lemma is true for systems having $n-1$ or less elements. Then $W_k(y_{i_1},y_{i_2},\ldots,y_{i_k}) > 0$ for $1 < i_1 < i_2 < \ldots < i_k \leq n$ since there are $n-1$ elements in $\{y_2,y_3,\ldots,y_n\}$ which set satisfies the hypothesis of the lemma and the induction. It must be shown that $W_k(y_{i_1},y_{i_2},\ldots,y_{i_k}) > 0$ when $i_1 = 1$.

If $k = 1$ then this is true. Therefore for $k > 1$ assume it is true for all smaller values of k. If $k = i_k$ then $W_k(y_{i_1},y_{i_2},\ldots,y_{i_k}) = W_k(y_1,\ldots,y_k) > 0$. When $i_k \neq k$ there is a smallest number j so that $i_j \neq j$. Assume the lemma is true for all larger values of j so that $W_k(y_1,\ldots,y_j,y_{i_{j+1}},\ldots,y_{i_{k-1}}) > 0$, in case $j = k$ this is true by hypothesis.

Now the last lemma implies that

$$W_{k-1}(y_2,\ldots,y_j,y_{i_j},\ldots,y_{i_{k-1}})W_k(y_{i_1},y_{i_2},\ldots,y_{i_{k-1}},y)$$

$$= W_{k-1}(y_{i_1},y_{i_2},\ldots,y_{i_{k-1}})W_k(y_2,\ldots,y_j,y_{i_j},y_{i_{j+1}},\ldots,y_{i_{k-1}},y)$$

$$+ W_{k-1}(y_{i_2},\ldots,y_{i_{k-1}},y)W_k(y_1,y_2,\ldots,y_j,y_{i_j},y_{i_{j+1}},\ldots,y_{i_{k-1}})$$

with y_j in the role of y_k of the previous lemma. Here however y_k is shifted to the left $k-j$ places in each Wronskian in which it occurs. Also $\{y_1,y_2,\ldots,y_{j-1},y_{i_j},\ldots,y_{i_k}\} = \{y_{i_1},y_{i_2},\ldots,y_{i_k}\}$ so that this set does not contain the y_j.

Now let $y = y_{i_k}$. Then $W_{k-1}(y_{i_2},\ldots,y_{i_{k-1}},y) > 0$ and $W_k(y_2,\ldots,y_j,y_{i_j},\ldots,y_{i_{k-1}},y) > 0$ by induction hypothesis on n. Further $W_k(y_{i_1},y_{i_2},\ldots,y_{i_{k-1}}) > 0$ by hypothesis on k and $W_k(y_1,\ldots,y_j,y_{i_j},\ldots,y_{i_{k-1}}) > 0$ by the last induction hypothesis on j. The realtion between the Wronskian's therefore imply that $W(y_{i_1},\ldots,y_{i_k}) > 0$ and the lemma is proved.

Relationships Between Principal Systems and Markov and Descartes Systems

The next lemma relates principal systems of solutions to Descartes systems. It is part b of Theorem 2.1 in Levin [4].

LEMMA 1.11 *If* $u_1,u_2,\ldots,u_n$ *is a principal system at* b *and* $W_k(u_1,u_2,\ldots,u_k) \neq 0$, $k = 1,2,\ldots,n$ *on a neighborhood of* b *then* $u_1,u_2,\ldots,u_n$ *is a Descartes system on a neighborhood of* b.

Proof. The proof is by induction on the number of functions n. If $n = 2$ then u_1, u_2 is a principal system of solutions at b implies $\lim_{t \to b^-} (u_2/u_1) = \infty$. Since $W_2(u_1,u_2) \neq 0$ on a neighborhood of b, $W_2(u_1,u_2)$ is of one sign on a neighborhood of b. If $W_2(u_1,u_2) < 0$ then since $W_2(u_1,u_2) = (u_2/u_1)' \, u_1^2$, (u_2/u_1) is a decreasing functions of a neighborhood of b. This contradicts that $\lim_{t \to b^-} (u_2/u_1) = \infty$ and so $W_2(u_1,u_2) > 0$.

For $n > 2$ suppose the lemma is true for all systems of less than n functions. Immediately, therefore, the system $\{u_1,u_2,\ldots,u_{n-1}\}$ is Descartes on a neighborhood of b. If it can be shown that $W_{n-1}(u_2,u_3,\ldots,u_n) \neq 0$ on a neighborhood of b, then the induction hypothesis will imply that $\{u_2,u_3,\ldots,u_n\}$ is a Descartes system on a neighborhood of b.

Suppose $W_{n-1}(u_2,u_3,\ldots,u_n) = 0$ on every neighborhood of b. Lemma 1.8 impies that

$$\left[\frac{W_{n-1}(u_2,u_3,\ldots,u_n)}{W_{n-1}(u_1,\ldots,u_{n-1})} \right]' = \frac{W_{n-2}(u_2,\ldots,u_{n-1})W_n(u_1,\ldots,u_n)}{(W_{n-1}(u_1,\ldots,u_{n-1}))^2}$$

and the Mean Value Theorem therefore implies that $W_{n-2}(u_2,\ldots,u_{n-1}) \times W_n(u_1,\ldots,u_n)$ is zero on every neighborhood of b. This is a contradiction and so $W_{n-1}(u_2,\ldots,u_n) \neq 0$ on some neighborhood of b. Since $\{u_1,u_2,\ldots,u_{n-1}\}$ is assumed to be Descartes, $W_{n-1}(u_2,\ldots,u_k) \neq 0$ $k = 2,3,\ldots,n$ on a neighborhood of b and so the induction hypothesis applies to $\{u_2,u_3,\ldots,u_n\}$ and it also is Descartes.

It remains to show that $W_n(u_1,u_2,\ldots,u_n) > 0$ and Lemma 1.10 will imply that since $W_{j-i+1}(u_i,u_{i+1},\ldots,u_j) > 0$, $i \leq i < j \leq n$, $\{u_1,u_2,\ldots,u_n\}$ is Descartes on a neighborhood of b. To show $W_n(u_1,u_2,\ldots,u_n) > 0$ one can consider $y_1 \equiv W_{n-1}(u_1,u_2,\ldots,u_{n-1})$ and

$y_2 \equiv W_{n-1}(u_1,u_2,\ldots,u_{n-2},u_n)$. The objective is to show that y_1,y_2 is a principal system at b and then use this fact to obtain a contradiction if $W_n(u_1,u_2,\ldots,u_n) < 0$ on a neighborhood of b.

The function y_1 is positive on a neighborhood of b by the hypothesis of the lemma. If y_2 is zero on every neighborhood of b then by Lemma 1.8

$$\left(\frac{y_2}{y_1}\right)' = \frac{W_{n-2}(u_1,\ldots,u_{n-2})W_n(u_1,\ldots,u_n)}{(W_{n-1}(u_1,\ldots,u_{n-1}))^2}$$

implies that $W_{n-2}(u_1,\ldots,u_{n-2})W_n(u_1,\ldots,u_n) = 0$ on every neighborhood of b, again by the Mean Value Theorem. This however is a contradiction and so $y_2 \neq 0$ on some neighborhood of b.

Since now $W_k(u_1,u_2,\ldots,u_k) \neq 0$ for $k = 1,2,\ldots,n-2$, the induction hypothesis applies to $\{u_1,u_2,\ldots,u_{n-2},u_n\}$ and so it is also Descartes. Therefore $W_{n-1}(u_1,u_2,\ldots,u_{n-2},u_n) > 0$ on a neighborhood of b. If $\{y_1, y_2\}$ is to be a principal system of solutions at b it is necessary that $\lim_{t \to b^-} (y_2/y_1) = \infty$. Since

$$\left(\frac{y_2}{y_1}\right)' = \frac{W_{n-2}(u_1,\ldots,u_{n-2})W_n(u_1,\ldots,u_n)}{(W(u_1,\ldots,u_{n-1}))^2}$$

is of constant sign in a neighborhood of b, (y_2/y_1) is monotonic and therefore $\lim_{t \to b^-} (y_2/y_1)$ exists. Since y_1 and y_2 are greater than zero in a neighborhood of b the limit is non-negative. Suppose the $\lim_{t \to b^-} (y_2/y_1)$ is other than ∞. Then if c is a large enough positive number, $(y_2/y_1) - c < 0$ on a neighborhood of b. Thus

$$W_{n-1}(u_1,\ldots,u_{n-2},u_n-cu_{n-1})$$
$$= W_{n-1}(u_1,u_2,\ldots,u_{n-2},u_n) - cW_{n-1}(u_1,u_2,\ldots,u_{n-1}) < 0,$$

on a neighborhood of b. However $\{u_1,u_2,\ldots,u_{n-2},u_n-cu_{n-1}\}$ is a principal system of solutions at b and it satisfies by its construction the induction hypothesis and therefore it is Descartes. This means that $W_{n-1}(u_1,u_2,\ldots,u_{n-2},u_n-cu_{n-1}) > 0$ on a neighborhood of b, which is a contradiction. This proves that $\{y_1,y_2\}$ is a principal system at b.

Now the facts that

$$\left(\frac{y_2}{y_1}\right)' = \frac{W_{n-2}(u_1,\ldots,u_{n-2})W_n(u_1,u_2,\ldots,u_n)}{W_{n-1}(u_1,\ldots,u_{n-1})^2}$$

and $\lim_{t\to b^-}(y_2/y_1) = \infty$ imply that $W_n(u_1,u_2,\ldots,u_n)$ cannot be negative everywhere on a neighborhood of b, and since by hypothesis $W_n(u_1,u_2,\ldots,u_n)$ has only one sign near b that sign can only be positive. This therefore concludes the proof of the lemma.

The main purpose of the lemmas up to this point is to prove the next group of theorems. Theorems 1.12 and 1.13 are basically Theorem 2.1 of Levin [4].

THEOREM 1.12 *If the* n^{th} *order linear differential operator* L *is disconjugate on* (a,b) *then there is a principal system of solutions of* $Lx = 0$ *at* b *and any such system is a Markov system on* (a,b) *and a Descartes system in a neighborhood of* b.

Proof. The operator L is disconjugate on (a,b) and therefore, as was proven in Lemma 1.5, it is also disconjugate on $(a,b]$. Further since L is disconjugate on (a,b), b is a nonoscillatory endpoint and Lemma 1.2

assures the existence of a principal system of solutions at b. Let such a principal system of solutions be $\{u_1,u_2,\ldots,u_n\}$. If $W(u_1,u_2,\ldots,u_k) = 0$ at some point t_o in (a,b) then since $W_k(u_1,\ldots,u_k)(t_o) = 0$ implies that the columns of the Wronskian matrix are linearly dependent there exists constants $c_1,c_2,\ldots,c_k$ not all zero such that the solution $u = c_1u_1+c_2u_2+\ldots+c_ku_k$ satisfies $Z_u(t_o) \geq k$. For this u, $Z_u(b) = \max\{\ell\colon \lim_{t\to b^-} (u(t)/u_{n-\ell+1}) = 0, \ell = 1,2,\ldots,n\}$ must be greater than or equal to $n-k$, since for $\ell \in [0,n-k]$, $\lim_{t\to b^-} (u/u_{n-\ell+1}) = 0$. Then $Z_u(a,b] = Z_u(a,b)+Z_u(b) \geq k+n-k = n$. The function u is a solution of $Lx = 0$ on (a,b) and L is disconjugate on $(a,b]$ so that $Z_u(a,b] < n$. Since the $u_1,\ldots,u_n$ are linearly independent u is not identically zero and so it cannot be that $W_k(u_1,u_2,\ldots,u_k) = 0$ anywhere on (a,b) for any $k = 1,2,\ldots,n$.

Now however the last lemma applies and so $\{u_1,u_2,\ldots,u_n\}$ is a Descartes system on a neighborhood of b. For a Descartes system $W_k(u_1,u_2,\ldots,u_k) > 0$, $k = 1,2,\ldots,n$ and since $W(u_1,\ldots,u_k)$ is a continuous function on (a,b) which is not zero on (a,b) it follows that $W_k(u_1,u_2,\ldots,u_k) > 0$ on all (a,b) for all $k = 1,2,\ldots,n$. That is $\{u_1,u_2,\ldots,u_n\}$ is a Markov system on (a,b). This completes this proof.

<u>THEOREM 1.13</u> *If* $u_1,u_2,\ldots,u_n$ *is a principal system of solutions at* b *for* $Lx = 0$ *and* L *is disconjugate on* (a,b) *then if* $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ *and* $1 \leq j_1 < j_2 < \ldots < j_k \leq n$ *are distinct sequences of integers such that* $i_\ell \leq j_\ell$, $\ell = 1,2,\ldots,k$ *then*

$$\lim_{t \to b^-} \frac{W_k(u_{i_1},u_{i_2},\ldots,u_{i_k})}{W_k(u_{j_1},u_{j_2},\ldots,u_{j_k})} = 0 .$$

Proof. Assume that $i_r < j_r$ for one r in $\{1,2,\ldots,k\}$ and that $i_\ell = j_\ell$ for $\ell \neq r$ this does not impair the generality of the theorem since if $i_{r_1} < j_{r_1}$ and $i_{r_2} < j_{r_2}$ and $i_\ell = j_\ell$ otherwise, then

$$\frac{W_k(u_{i_1},\ldots,u_{i_{r_1}},\ldots,u_{i_{r_2}},\ldots,u_{i_k})}{W_k(u_{j_1},\ldots,u_{j_{r_1}},\ldots,u_{j_{r_2}},\ldots,u_{j_k})}$$

$$= \frac{W_k(u_{i_1},\ldots,u_{i_{r_1}},\ldots,u_{j_{r_2}},\ldots,u_{i_k})}{W_k(u_{j_1},\ldots,u_{j_{r_1}},\ldots,u_{j_{r_2}},\ldots,u_{j_k})} \times \frac{W_k(u_{i_1},\ldots,u_{i_{r_1}},\ldots,u_{i_{r_2}},\ldots,u_{i_k})}{W_k(u_{j_1},\ldots,u_{i_{r_1}},\ldots,u_{j_{r_2}},\ldots,u_{j_k})} .$$

Since $\{u_1,u_2,\ldots,u_n\}$ is a principal system and $i_r < j_r$ $\lim_{t \to b} (u_{i_r})/(u_{j_r}) = 0$ and therefore for $c \in (0,\infty)$ $u_{j_r} - cu_{i_r} > 0$ on a neighborhood of b which of course depends on c. Fix c. Let $w_{j_r} = u_{j_r} - cu_{i_r}$ and then $\{u_1,\ldots,u_{j_{(r-1)}},w_{j_{r_1}}, u_{j_{(r+1)}},\ldots,u_n\}$ is a principal system at b for $Lx = 0$. Since L is disconjugate on (a,b) Theorem 1.12 implies that $W_k(u_{i_1},\ldots,u_{i_{(r-1)}}, w_{j_r}, u_{i_{(r+1)}}, \ldots, u_{i_k}) > 0$ on a neighborhood of b; it must be remembered that $i_\ell = j_\ell$ for $\ell \neq r$. This may be written as

$$W_k(u_{i_1},\ldots,u_{i_{(r-1)}},u_{i_r},u_{i_{(r+1)}},\ldots,u_{i_k})$$

$$- c\, W_k(u_{i_1},\ldots,u_{i_{(r-1)}},u_{i_r},u_{i_{(r+1)}},\ldots,u_{i_k}) > 0$$

on a neighborhood of b. Thus for each $c \in (0,\infty)$ there is a neighborhood of b so that

$$(W_k(u_{j_1},\ldots,u_{j_k}))/(W_k(u_{i_1},\ldots,u_{i_k})) > c \quad ,$$

therefore $\lim_{t \to b^-} (W_k(u_{j_1},\ldots,u_{j_k})/W_k(u_{i_1},\ldots,u_{i_k})) = \infty$ and so the theorem is proved.

The concept of a principal system solutions was defined for both endpoints of an interval (a,b). In the propositions up to this point the left endpoint a has been, for the most part, ignored. The methods of proof are, however, of such a nature as to apply equally well at a as at b. Therefore if L is disconjugate on (a,b) there are principal systems of solutions of $Lx = 0$ at a which are all Descartes on neighborhoods of a and which are all Markov on (a,b). These principal systems of solutions are of the form $u_1,u_2,\ldots,u_n$ so that $\lim_{t \to a^+} (u_{k+1}/u_k) = \infty$ and $u_i > 0$ on a neighborhood of a.

Fundamental Principal Systems

In connection with principal systems of solutions at both endpoints of an interval (a,b) Levin in [4] introduced the concept of a Hierarchical Fundamental System. Willett in [10] called such a system a Fundamental Principal System and it is this terminology that is used here following the example of Muldowney in [5].

A Fundamental Principal system of solutions of $Lx = 0$ on an interval (a,b) is a set $\{u_1,u_2,\ldots,u_n\}$ of solutions of $Lx = 0$ such that $u_1,u_2,\ldots,u_n$ is a principal system of solutions at b and $u_n,u_{n-1},\ldots,u_1$ is a principal system at a. The reverse ordering of the

functions at a means that $Z_f(a)$ is $\max\{\ell : \lim_{t \to a^+} (f/u_\ell) = 0\}$.

In this connection the next theorem is, except for those parts which have already appeared in Theorems 1.12 and 1.13, Lemma 4.1 of Levin [4].

THEOREM 1.14 *If an n^{th} order linear differential operator L is disconjugate on $[a,b]$ then there exists a fundamental principal system of solutions $u_1,u_2,\dots,u_n$ on* (a,b). *In addition this system is Descartes on (a,b) and $\lim_{t \to a^+(b^-)} (W_k(u_{i_1},\dots,u_{i_k})/W_k(u_{j_1},\dots,u_{j_k})) = 0(\infty)$ if $j_r \le i_r$ for all r and $j_\ell < i_\ell$ for some ℓ, $r,\ell \in \{1,2,\dots,k\}$.*

Proof. The solution space of the equation $Lx = 0$ has dimension n since L is an n^{th} order linear operator. Let $x_1,x_2,\dots,x_n$ be a principal system of solutions for $Lx = 0$ at b and let $y_n,y_{n-1},\dots,y_1$ be a principal system of solutions for $Lx = 0$ at a. For any k, $1 \le k \le n$, every element in the space spanned by $x_1,x_2,\dots,x_k$ has at least $n-k$ zeros at b. Every element in the space spanned by $\{y_n,y_{n-1},\dots,y_k\}$ has at least $k-1$ zeros at a. If the intersection of these two spaces is the trivial solution then the sum of the two spaces is a subspace with basis $\{x_1,x_2,\dots,x_k,y_k,\dots,y_n\}$, and dimension $n+1$, of the space of solutions of $Lx = 0$. Therefore the intersection cannot be the trivial solution. Let u_k be an element in that intersection, $u_k \not\equiv 0$. Then $Z_{u_k}(a) \ge k-1$ and $Z_{u_k}(b) \ge n-k$. Since L is disconjugate on $[a,b]$ these inequalities can only be equalities. Further since u_k can have no zero in (a,b) it can be choosen to be positive on all of (a,b).

This set $u_1,u_2,\dots,u_n$ so choosen is a fundamental principal

system on (a,b).

The u_k are unique up to a constant multiple. This is so because if u_k and x_k are solutions such that $Z_{u_k}(a) = k-1 = Z_{x_k}(a)$ and $Z_{u_k}(b) = n-k = Z_{x_k}(b)$ and u_k and x_k are positive on (a,b) then letting $x(t) = u_k(s)x_k(t)-u_k(t)x_k(s)$ for some s in (a,b) gives another solution of $Lx = 0$. This solution $x(t)$ has a zero at s, $k-1$ zeros at a and $n-k$ zeros at b. This is a total of n zeros on $[a,b]$ and the disconjugacy of L implies that $x(t)$ is identically equal to zero on (a,b).

To show the fundamental principal system of solutions is Descartes, making use of Lemma 1.10 it is only necessary to show that $W_{k+1}(u_i,u_{i+1},\ldots,u_{i+k}) > 0$ for all $1 \leq i \leq n$ and $k \in \{0,1,\ldots,n-i\}$. Suppose for a contradiction that $W_{k+1}(u_i,u_{i+1},\ldots,u_{i+k}) \leq 0$ somewhere in (a,b). Since $W_{k+1}(u_i,u_{i+1},\ldots,u_{i+k})$ is a continuous function and since it is positive on neighborhoods of a and b it must be zero in (a,b). Suppose therefore that there is an $s \in (a,b)$ such that $W_{k+1}(u_i,u_{i+1},\ldots,u_{i+k})(s) = 0$. The rows of the Wronskian matrix evaluated at s are therefore linearly dependent and so there exist constants $c_0,c_1,\ldots,c_k$ not all zero such that $\sum_{j=0}^{k} c_j\, u_{i+j}^{(\ell)}(s) = 0$ for $\ell = 0,1,\ldots,k$. Let $x(t) = \sum_{j=0}^{k} c_j\, u_{i+j}$ and then $Z_x(s) \geq k+1$. Also $Z_x(a) \geq i-1$ and $Z_x(b) \geq n-(i+k)$. The total number of zeros of x on $[a,b]$ is therefore greater than or equal to n and so the disconjugacy of L implies that $x(t) \equiv 0$ on (a,b). This in turn implies that $u_i,u_{i+1},\ldots,u_{i+k}$ are a linearly dependent set of functions on (a,b) which implies that $W_{k+1}(u_i,u_{i+1},\ldots,u_{i+k}) \equiv 0$ on (a,b) which contradicts

that the Wronskian is strictly positive on neighborhoods of a and b.

The part of the theorem involving the limit of the quotient of Wronskians is proven in Theorem 1.13 and this observation concludes the proof.

This last theorem provides for the existence of a fundamental principal system of solutions which is Descartes on (a,b) if L is disconjugate on $[a,b]$. To get the converse of this the next lemma which is part of Proposition 5, Chapter III of Coppel's book [1] is useful.

LEMMA 1.15. *If* $u_1,u_2,\ldots,u_n$ *are* n *functions in* $C^n(a,b)$ *and if the Wronskians* $W_{j+1}(u_i,u_{i+1},\ldots,u_{i+j})$, $i = 1,2,\ldots,n$; $j \in \{0,1,2,\ldots,n-i\}$ *do not vanish on* (a,b) *then the maximum number of zeros that an arbitrary nontrivial linear combination* $u = c_iu_i + \ldots + c_{i+k}u_{i+k}$ *can have is* k.

Proof. The proof of this is by induction on the number n. For $n = 1$, $u_1 \in C^1(a,b)$ and $c_1u_1 = 0$ at t_o only if $u_1(t_o) = 0$ if $c_1 \neq 0$.

Suppose then that the lemma is true for all systems of $n-1$ functions in $C^{n-1}(a,b)$. Then for $k = 0$ c_iu_i has no zeros on (a,b) unless u_i has a zero if $c_i \neq 0$. For $k \geq 1$ suppose there is a linear combination $c_iu_i + \ldots + c_{i+k}u_{i+k} \equiv u$ which has $k+1$ zeros in (a,b). Then (u/u_i) has $k+1$ zeros on (a,b) and by Rolle's theorem between consecutive zeros and Leibnitz's formula $(u/u_i)^{(m)} = \sum_{p=0}^{m} u^{(m-p)}(1/u_i)^{(p)}$ at multiple zeros, $(u/u_i)'$ has k zeros on (a,b).

The functions $(u_1/u_i)'$, $(u_2/u_i)'$, $\ldots$, $(u_{i-1}/u_i)'$, $(u_{i+1}/u_i)'$,

..., $(u_n/u_i)'$ are $n-1$ functions in $C^{n-1}(a,b)$, and by Corollary 1.17, $W_k((u_{i+1}/u_i)',\ldots,(u_{i+k}/u_i)') = (u_i)^{k+1}\, W_{k+1}(u_i,u_{i+1},\ldots,u_{i+k}) \neq 0$ on (a,b). The induction hypothesis implies that a nonzero linear combination of $(u_{i+1}/u_i)',\ldots,(u_{i+k}/u_i)'$ has at most $k-1$ zeros in (a,b). Therefore the linear combination $c_{i+1}(u_{i+1}/u_i)'+\ldots+c_{i+k}(u_{i+k}/u_i)' = (u/u_i)'$ has at most $k-1$ zeros in (a,b). This contradicts that $(u/u_i)'$ has k zeros there and so the lemma is proved.

The next theorem appears in the proof of Theorem 4.1 in Levin [4].

THEOREM 1.16 *If* $Lx = 0$ *has a fundamental principal system of solutions on* (a,b) *which is also Descartes there, then* L *is disconjugate on* $[a,b]$.

Proof. Let u be a solution of $Lu = 0$. Let $u_1,u_2,\ldots,u_n$ be the system given in the hypothesis of the theorem. Then $u = \sum_{k=i}^{j+1} c_k u_k$ with $c_i \neq 0$ and $c_{i+j} \neq 0$. Therefore $Z_u(a) = i-1$ and $Z_u(b) = n-(i+j)$. Since $W_{j+1}(u_i,u_{i+1},\ldots,u_{i+j}) \neq 0$ on (a,b) u has at most j zeros in (a,b) by the last lemma. Therefore for every $u \in C^n(a,b)$ so that $Lu = 0$ on (a,b); $Z_u[a,b] \leq (i-1) + n-(i+j) + j = n-1$. This is just the statement that L is disconjugate on $[a,b]$ and the theorem is proved.

Generalized Derivatives

When L is disconjugate on $[a,b]$ it becomes possible to define a generalized derivative which gives the number of generalized zeros of a

function f with respect to L in the same way the usual derivatives of f give the usual zeros of f on (a,b). The generalized derivative need only be defined at a and b when they are singular endpoints.

Two such generalized derivatives have been suggested. Willett introduced the idea in [10]. His definition gives a different definition for the generalized zeros of a function which is not a solution of $Lx = 0$. If $u_1, u_2, \ldots, u_n$ is a fundamental principal system on (a,b) for L Willett defines the j^{th} derivative of f at a by

$$\mathcal{D}^{(j)} f(a) = \lim_{t \to a^+} \frac{W_{j+1}(u_1, \ldots, u_j, f)}{W_{j+1}(u_1, \ldots, u_j, u_{j+1})}$$

and at b by

$$\mathcal{D}^{(j)} f(b) = \lim_{t \to b^-} \frac{W_{j+1}(u_n, \ldots, u_{n-j+1}, f)}{W_{j+1}(u_n, \ldots, u_{n-j})} .$$

J.S. Muldowney in [5] in order to keep Levin's definition of generalized zero suggested another definition for a derivative. He defines the j^{th} derivative of f at a by $\mathcal{D}_\ell^{(0)} f(a) = \lim_{t \to a^+} (f/u_1)$,

$$\mathcal{D}_\ell^{(j)} f(a) = \lim_{t \to a^+} \left[\frac{f - \sum_{i=0}^{j-1} u_{i+1} \mathcal{D}_\ell^{(i)} f(a)}{u_{j+1}} \right], \quad j > 0$$

and at b by $\mathcal{D}_\ell^{(0)} f(b) = \lim_{t \to b^-} (f/u_n)$,

$$\mathcal{D}_\ell^{(j)} f(b) = \lim_{t \to b^-} \left[\frac{f - \sum_{i=0}^{j-1} u_{n-i} \mathcal{D}_\ell^{(i)} f(b)}{u_{n-j}} \right], \quad j > 0 .$$

In both cases j is in $\{0,1,\ldots,n-1\}$. Muldowney in [5] investigated the relation between these two derivatives. He defined $\bar{\mathcal{D}}^{(j)} f(a)$ by

$$\bar{\mathcal{D}}^{(j)}f(a) = \limsup_{a^+} \frac{W_{j+1}(u_1,\ldots,u_j\ f)}{W_{j+1}(u_1,\ldots,u_{j+1})}$$

and $\underline{\mathcal{D}}^{(j)}f(a) = -\bar{\mathcal{D}}^{(j)}(-f(a))$; $\bar{\mathcal{D}}^{(j)}f(b)$ is defined by

$$\bar{\mathcal{D}}^{(j)}f(b) = \limsup_{b^-} \frac{W_{j+1}(u_n,\ldots,u_{n-j+1},f)}{W_{j+1}(u_n,\ldots,u_{n-j})}$$

and $\underline{\mathcal{D}}^{(j)}f(b) = -\bar{\mathcal{D}}^{(j)}(-f(b))$, $j = 0,1,\ldots,n-1$.

To deal with these derivatives he proved the next lemma which is found on page 94 in [5].

LEMMA 1.17 *Let* F *and* G *be two real functions differentiable on a neighborhood of* t. *Let* $\{r_n\}$ *and* $\{s_n\}$ *be real sequences such that* $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = t$, *with* $s_n \neq t$ *and* $r_n \neq t$ *for all* n. *Further let* $\lim_{n\to\infty} (F(r_n)/G(r_n)) = \ell$ *and either*

i) $\lim_{n\to\infty} F(s_n) = \lim_{n\to\infty} G(s_n) = 0$

or

ii) $\lim_{n\to\infty} G(r_n) = \infty$.

Then with these hypotheses there is a sequence $\{t_n\}$ *such that* $\lim_{n\to\infty} t_n = t$ *and* $\lim_{n\to\infty} (F'(t_n)/G'(t_n)) = \ell$.

Proof. (Case i) For a fixed N the Cauchy Mean Value Theorem gives that $(F'(c_N)/G'(c_N)) = (F(r_N)-F(s_n))/(G(r_N)-G(s_n))$ for some c_N between r_N and s_n. Since

$$\lim_{n\to\infty} \frac{F(r_N)-F(s_n)}{G(r_N)-F(s_n)} = \frac{F(r_N)}{G(r_N)}$$

there are points c_N between r_N and t with $(F'(c_N)/G'(c_N))$ arbitrarily close to $(F(r_N)/G(r_N))$ for each N. This proves i).

(Case ii) For some c_N between r_N and r_n

$$\frac{F'(c_N)}{G'(c_N)} = \frac{F(r_n) - F(r_N)}{G(r_n) - G(r_N)}$$

$$= \left[\frac{F(r_n)}{G(r_n)} - \frac{F(r_N)}{G(r_N)} \right] \left[1 - \frac{G(r_N)}{G(r_n)} \right]^{-1} .$$

The limit of the last is ℓ so that the lemma is proved for case ii).

In the next proposition it is necessary to distinguish which fundamental principal system is used in computing the generalized derivatives. To this end if $S = \{u_1, u_2, \ldots, u_n\}$ then the derivatives calculated with reference to S are written $S\,\mathcal{D}^{(j)} f(t)$.

This next proposition is Proposition 3.2 of [5].

<u>PROPOSITION 1.18</u> *If* $S = \{u_1, u_2, \ldots, u_n\}$ *is a fundamental principal system of solutions on* (a,b) *and if* $\mathcal{D}_\ell^{(j)} f(a) = \lambda$ *then* $\underline{\mathcal{D}}^{(j)} f(a) \leq \lambda \leq \bar{\mathcal{D}}^{(j)} f(a)$. *The same inequalities hold true at* b.

<u>Proof</u>. The derivatives $\mathcal{D}_\ell^{(k)} f(a)$, $k = 0,1,\ldots,j-1$ exist and are finite. Assume $\mathcal{D}_\ell^{(k)} f(a) = 0$ for $k < j$. This is replacing f by $f - \sum_{i=0}^{j-1} u_{i+1} \mathcal{D}_\ell^{(j)} f(a)$, but

$$W_{j+1}(u_1, \ldots, u_j, f) = W_{j+1}(u_1, \ldots, u_j, f - \sum_{i=0}^{j-1} u_{i+1}\, \mathcal{D}_\ell^{(j)} f(a)) .$$

The proposition is true if $j = 0$, since $\mathcal{D}_{\ell}^{(0)} f(a) = \mathcal{D}^{(0)} f(a) = \lim_{t \to a^+} (f/u_1)$. For an induction assume it is true for $i \leq j-1$ so that $\underline{\mathcal{D}}^{(i)} f(\alpha) \leq 0 \leq \overline{\mathcal{D}}^{(i)} f(a)$ for $i = 0,1,\ldots,j-1$. Therefore there is a sequence $\{s_k\}$ with $\lim_{t \to \infty} s_k = a$ and

$$\lim_{k \to \infty} \frac{W_j(u_1,\ldots,u_{j-1},f)}{W_j(u_1,\ldots,u_j)} (s_k) = 0 \quad .$$

Let $S_o = \{u_1,u_2,\ldots,u_{j-1},u_{j+1}\}$. S_o is a fundamental principal system of solutions on (a,b) and

$$S_o\ \mathcal{D}_{\ell}^{(i)} f(a) = 0 \ , \quad i = 0,\ldots,j-2 \ .$$

$$S_o\ \mathcal{D}_{\ell}^{(j-1)} f(a) = \lambda \ ,$$

so the induction hypothesis gives a sequence $\{r_k\}$ with $\lim_{k \to \infty} r_k = a$ and

$$\lim_{k \to \infty} \frac{W_j(u_1,\ldots,u_{j-1},f)}{W_j(u_1,\ldots,u_{j-1},u_{j+1})} (r_k) = \lambda \quad .$$

Let

$$F = \frac{W_j(u_1,\ldots,u_{j-1},f)}{W_j(u_1,\ldots,u_j)}$$

and

$$G = \frac{W_j(u_1,\ldots,u_{j-1},u_{j+1})}{W_j(u_1,\ldots,u_j)} \quad .$$

Then

$$\frac{F'}{G'} = \frac{W_{j+1}(u_1,\ldots,u_j,f)}{W_{j+1}(u_1,\ldots,u_j,u_{j+1})} \quad .$$

This last quotient converges to λ on a sequence $\{t_k\}$ with $\lim_{k \to \infty} t_k = a$ by the last lemma. Therefore

$$S\, \underline{\mathcal{D}}^{(j)} f(a) \leq S\, \mathcal{D}_{\ell}^{(j)} f(a) \leq S\, \overline{\mathcal{D}}^{(j)} f(a) \quad .$$

This concludes the proof.

CHAPTER II

ABSTRACT

The purpose of the second chapter is basically to prove the theorem of Muldowney mentioned in the abstract by a method which he did not use, but which he suggested in the paper in which the theorem appeared. The theorem itself is a generalization of the fact that a function with a positive derivative on an interval is increasing on the interval. The theorem itself has its origin in a paper of Pólya from 1922 where it appeared in a somewhat different version without consideration of singular points.

In this chapter the original theorem of Pólya is first given recast in the language of Muldowney. Next the changes necessary to the proof of Pólya's theorem required by the introduction of singular end-points and generalized zeros are prepared and finally the generalized form of the theorem is proved. The chapter finishes with the presentation of some examples and with Pólya's Mean Value Theorem in the generalized sense.

CHAPTER II

A PROOF OF PÓLYA'S THEOREM

Background

In [7], Pólya gave a Mean Value Theorem involving a linear differential operator. Willett [10] and Muldowney [5] extended this theorem to include the case of singular points of the operator. Polya gave two proofs. This first is done in detail and a second proof is just sketched. In Muldowney [5] there are also two proofs. One is given in detail and the other is simply outlined. The proof outlined is the extension to include singular points of the proof sketched by Pólya. This chapter is basically a proof of the mean value theorem as formulated by Muldowney in [5] and using this alternative method of proof.

Pólya's Theorem

Let (a,b) be an interval of real numbers. Let $a \leq t_1 < t_2 < \dots < t_m \leq b$ be m points in $[a,b]$ and let $r_1, r_2, \dots, r_m$ be m positive integers whose sum is n. Define $W_n(u_1,u_2,\dots,u_n)\begin{bmatrix} t_1,t_2,\dots,t_n \\ r_1,r_2,\dots,r_m \end{bmatrix}$ to be the determinant of the matrix with k^{th} row

$$(u_k(t_1),u_k'(t_1),\dots,u_k^{(r_1-1)}(t_1),u_k(t_2),u_k'(t_2),\dots,u_k^{(r_2-1)}(t_2),$$
$$\dots,u_k(t_m),u_k'(t_m),\dots,u_k^{(r_m-1)}(t_m)), \quad k = 1,2,\dots,n .$$

If $t_1 = a$ or $t_m = b$ and if $u_1,u_2,\dots,u_{n-1}$ is a fundamental principal system of solutions on (a,b) then replace $u_k^{(j)}(a)$ by

$\mathcal{D}^{(j)}u_k(a)$ and $u_k^{(j)}(b)$ by $\mathcal{D}^{(j)}u_k(b)(-1)^j$.

The first theorem is Polya's theorem although its present form is due to J.S. Muldowney. In order to prove the theorem a preliminary lemma is useful.

<u>LEMMA 2.1</u> *Let* $u_1, u_2, \ldots, u_n$ *be a Markov system of functions on* (a,b) *and let* $a < t_1 < t_2 < \ldots < t_m < b$ *and let* $r_1, r_2, \ldots, r_m$ *be positive integers with* $\sum_{i=1}^{m} r_i = n+1$. *Then*

$$W_{n+1}(u_1, u_2, \ldots, u_n, u_{n+1})\begin{bmatrix} t_1, & t_2, & \ldots, & t_m \\ r_1, & r_2, & \ldots, & r_m \end{bmatrix}$$

$$= \prod_{i=1}^{m} (u_1(t_i))^{r_i} \int_{t_1}^{t_2} \int_{t_2}^{t_3} \cdots \int_{t_{m-1}}^{t_m} W_n\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_n}{u_1}\right)', \left(\frac{u_{n+1}}{u_1}\right)'\right)$$

$$\begin{bmatrix} t_1, & s_1, & t_2, & s_2, & \ldots, & s_{m-1}, & t_m \\ r_1-1, & 1, & r_2-1 & 1, & \ldots, & 1, & r_m-1 \end{bmatrix}$$

$$ds_{m-1}, ds_{m-2} \ldots ds_1 .$$

<u>Proof</u>. This proof uses Leibnitz's rule

$$(u_k/u_1)^{(m)} = \sum_{p=0}^{m} \binom{m}{p} u_k^{(m-p)} (1/u_1)^{(p)} .$$

In the matrix with determinant $W_{n+1}(u_1, \ldots, u_n, f)\begin{bmatrix} t_1, & t_2, & \ldots, & t_m \\ r_1, & r_2, & \ldots, & r_m \end{bmatrix}$, if the columns evaluated at t_i are divided by $u_1(t_i)$, one obtains the matrix with k^{th} row

$$\left(\frac{u_k}{u_1}(t_1), \ldots \frac{u_k'(t_1)}{u_1(t_1)}, \ldots, \frac{u_k^{(r_1-1)}(t_1)}{u_1(t_1)}, \frac{u_k(t_2)}{u_1(t_2)}, \ldots, \right.$$

$$\left. \frac{u_k^{(r_2-1)}(t_2)}{u_1(t_2)}, \ldots, \frac{u_k(t_m)}{u_1(t_m)}, \ldots \frac{u_k^{(r_m-1)}(t_m)}{u_1(t_m)} \right),$$

$k = 1,2,\ldots,n+1$. Now Leibnitz's formula allows this matrix to be changed by adding suitable multiples of columns to other columns into the matrix with k^{th} row

$$\left(\frac{u_k}{u_1}(t_1), \left(\frac{u_k}{u_1}\right)'(t_1), \ldots, \left(\frac{u_k}{u_1}\right)^{(r_1-1)}(t_1), \ldots, \left(\frac{u_k}{u_1}\right)(t_m), \left(\frac{u_k}{u_1}\right)'(t_m), \right.$$

$$\left. \ldots \left(\frac{u_k}{u_1}\right)^{(r_m-1)}(t_m) \right)$$

without changing the value of the determinant. This is done starting on the right hand side and moving left. For example the $n+1^{st}$ column is obtained from the sums

$$\begin{vmatrix} 0 \\ \frac{u_2}{u_1}^{(r_m-1)}(t_m) \\ \cdot \\ \cdot \\ \cdot \\ \frac{u_{n+1}}{u_1}^{(r_m-1)}(t_m) \end{vmatrix} = \sum_{k=0}^{m} \binom{m}{p} \begin{vmatrix} u_1^{(m-p)}(t_m) \\ u_2^{(m-p)}(t_m) \\ \cdot \\ \cdot \\ \cdot \\ u_{n+1}^{(m-p)}(t_m) \end{vmatrix} \left(\frac{1}{u_1}\right)^{(p)}(t_m)$$

where $m = r_m-1$. Next subtracting the solumns $(1, (u_2/u_1)(t_j), (u_3/u_1)(t_j), \ldots, (u_n/u_1)(t_j), (u_{n+1}/u_1)(t_j))^T$ from $(1, (u_2/u_1)(t_{j+1}), (u_3/u_1)(t_{j+1}), \ldots, (u_n/u_1)(t_{j+1}), (u_{n+1}/u_1)(t_{j+1}))^T$ for $j = m-1$ and then $j = m-2$ and so on along to $j = 1$ gives the matrix with k^{th} row

$$((u_k/u_1)(t_1),\ (u_k/u_1)'(t_1),\ \ldots,\ (u_k/u_1)^{(r_1-1)}(t_1),\ (u_k/u_1)(t_2) - (u_k/u_1)(t_1),$$
$$(u_k/u_1)'(t_2),\ \ldots,\ (u_k/u_1)^{(r_2-1)}(t_2),\ (u_k/u_1)(t_3) - (u_k/u_1)(t_2),$$
$$(u_k/u_1)'(t_3),\ \ldots,\ (u_k/u_1)^{(r_m-1)}(t_m)) \quad k = 1,2,\ldots,n+1 .$$

The first row of this matrix is just $(1,0,\ldots,0)$. The expansion of this determinant along the first row gives the same thing one gets if one evaluates the integral in the statement of the lemma. This is because the integration variable occurs in only one column and so the integration may be moved inside the matrix and the integration done to the column containing the variable before the determinant is evaluated. The justification for this is that expansion of the determinant along the column containing the integration variable gives the determinant to be just a linear combination of the elements in that column the coefficients of which are constant with respect to the integration variable.

The product $\prod_{i=1}^{m} (u_1(t_i))^{r_i}$ arises from the initial dividing of the columns of the matrix by $u_1(t_i)$.

This concludes the proof of the lemma.

<u>THEOREM 2.2</u> *Let* $u_1,u_2,\ldots,u_n$ *be an ordered set of* n *times differentiable functions on* (a,b) *and let* $a < t_1 < t_2 < \ldots < t_m < b$ *be* m *points of* (a,b) *and* $r_1,r_2,\ldots,r_m$, m *positive integers with* $\sum_{i=1}^{m} r_i = n+1$. *Then if* f *is* n *times differentiable on* (a,b) *and if*

i) $W_{n+1}(u_1,u_2,\ldots,u_n,f) \geq 0$ *on* (t_1,t_m)

and

ii) $W_k(u_1,\ldots,u_k) > 0$ *on* (a,b) $\quad k = 1,2,\ldots,n$

then

$$W_{n+1}(u_1,u_2,\ldots,u_n,f)\begin{bmatrix} t_1, & t_2, & \ldots, & t_m \\ r_1, & r_2, & \ldots, & r_m \end{bmatrix} \geq 0 \quad ,$$

and if strict inequality holds for any t *in* (t_1,t_m) *in* i *then strict inequality holds in the conclusion.*

If instead of f *being* n*-times differentiable,* f *is* loc $AC^{n-1}(a,b)$ *and the strict inequality holds on a set of measure greater than zero the theorem is also true.*

<u>Proof</u>. In this theorem t_1 and t_m are not allowed to be the endpoint of the interval (a,b) and so it is not necessary that $u_1,u_2,\ldots,u_n$ be a fundamental principal system of solutions on (a,b) in order that $W_{n+1}(u_1,\ldots,u_n,f)\left[\begin{smallmatrix} t_1,\ldots,t_m \\ r_1,\ldots,r_m\end{smallmatrix}\right]$ be defined.

The proof is by an induction on n, the order of the set $\{u_1,\ldots,u_n\}$.

For $n = 1$, the conditions of the theorem are $W_2(u_1,f) \geq 0$ on (t_1,t_2) and $u_1 > 0$ on (a,b). That $u_1 > 0$ on (a,b) allows that (f/u_1) is well defined and differentiable on (a,b). The fact that $(f/u_1)'\, u_1^2 = W(u_1,f) \geq 0$ implies that (f/u_1) is a nondecreasing function on (t_1,t_2) and if (f/u_1) is not constant, that is if $W(u_1,f) > 0$ anywhere in (t_1,t_2), then $(f/u_1)(t_2) - (f/u_1)(t_1) > 0$. But $(f/u_1)(t_2) - (f/u_1)(t_1) = u_1(t_1)u_1(t_2)W(u_1,f)\left[\begin{smallmatrix} t_1 & t_2 \\ 1 & 1\end{smallmatrix}\right]$ and so the theorem is true for $n = 1$. In the case $f \in$ loc $AC^{n-1}(a,b)$ the fact that

$$\frac{f}{u_1}(t_2) - \frac{f}{u_1}(t_1) = \int_{t_1}^{t_m} \left(\frac{f}{u_1}\right)' dt$$

is used instead.

Suppose now for $n > 1$ that the theorem is true for all systems or order less than n. Then by Lemma 2.1

$$W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} t_1, & \dots, & t_m \\ r_1, & \dots, & r_m \end{bmatrix} = \prod_{i=1}^{m} (u_1(t_i))^{r_i}$$

$$\times \int_{t_1}^{t_2}\int_{t_2}^{t_3}\cdots\int_{t_{m-1}}^{t_m} W_n\left(\left(\frac{u_2}{u_1}\right)', \dots, \left(\frac{u_n}{u_1}\right)', \left(\frac{f}{u_1}\right)'\right)$$

$$\times \begin{bmatrix} t_1, & s_1, & t_2, & s_2, & \dots, & s_{m-1}, & t_m \\ r_1-1, & 1, & r_2-1, & 1, & \dots, & 1, & r_m-1 \end{bmatrix} d_{m-1}\; ds_{m-2}\; \cdots\; ds_1.$$

Now by Lemma 1.7 since $u_1,\dots,u_n$ is Markov on (a,b), $(u_2/u_1)'$, ..., $(u_n/u_1)'$ is Markov on (a,b) and by the same lemma $W_n((u_2/u_1)', \dots, (u_n/u_1)', (f/u_1)') \geq 0$ if $W_{n+1}(u_1,\dots,u_n,f) \geq 0$, and so the induction hypothesis implies that the integrand

$$W_n((u_2/u_1)',\dots,(u_n/u_1)',(f/u_1)')\begin{bmatrix} t_1, & s_1, & t_2, & \dots, s_{m-1}, & t_m \\ r_1-1, & 1, & r_2-1, \dots, & 1, & r_m-1 \end{bmatrix} \geq 0$$

and so the integral is non-negative. If strict inequality holds in i) of the hypothesis then the strict inequality also holds in $W_n((u_2/u_1)',\dots,(u_n/u_1)',(f/u_1)') \geq 0$ and so the induction gives that the integrand is strictly positive on a subset of positive measure of the set $[t_1,t_2] \times \dots \times [t_{m-1},t_m]$ and therefore the integral is strictly positive. This last statement is true whether f is n times differentiable or f is loc $AC^{n-1}(a,b)$.

This concludes the proof of the theorem.

Some Facts About Generalized Derivatives

Essentially the same proof works in the case that t_1 or t_m is an endpoint of (a,b). However in this case the generalized derivatives enter and it is necessary to have some more information on these generalized derivatives in order to proceed.

LEMMA 2.3 *Let* $u_1,u_2,\ldots,u_n$ *be an ordered set of* n *times differentiable functions on* (a,b) *and let* f *be an* n *times differentiable function on* (a,b). *Then if* $W_k(u_1,\ldots,u_k)$, $k = 1,2,\ldots,n$ *and* $W_{n+1}(u_1,u_2,\ldots,u_n,f)$ *do not change sign on a neighborhood of* a

$$\lim_{t \to a^+} \frac{W_{j+1}(u_1,u_2,\ldots,u_j,f)}{W_{j+1}(u_1,u_2,\ldots,u_{j+1})} \; ; \qquad j = 0,1,\ldots,n-1$$

exists and if $W_{k+1}(u_n,\ldots,u_{n-k})$, $k = 0,1,\ldots,n-1$ *and* $W_{m+1}(u_1,u_2,\ldots,u_n,f)$ *do not change sign on a neighborhood of* b

$$\lim_{t \to b^-} \frac{W_{j+1}(u_n,\ldots,u_{n-j+1},f)}{W_{j+1}(u_n,\ldots,u_{n-j})} \; ; \quad j = 0,1,\ldots,n-1$$

exists.

Proof. The proof is an induction on $n-j$, throughout n is constant and fixed. For $n-j = 1$ one has

$$\left[\frac{W_n(u_1,\ldots,u_{n-1},f)}{W_n(u_1,\ldots,u_n)} \right]' = \frac{W_{n-1}(u_1,\ldots,u_{n-1})W_{n+1}(u_1,\ldots,u_n,f)}{(W_n(u_1,\ldots,u_n))^2}$$

However $W_{n-1}(u_1,\ldots,u_{n-1})$ and $W_{n+1}(u_1,\ldots,u_n,f)$ do not change sign on a neighborhood of a and so $[(W_n(u_1,\ldots,u_{n-1},f)/W_n(u_1,\ldots,u_n)]$ is a monotonic function and therefore the limit in question exists.

Assume the lemma is true for all $n-j < k$ and show it is true for $n-j = k$. One has

$$\left[\frac{W_{n-k+1}(u_1,\ldots,u_{n-k},f)}{W_{n-k+1}(u_1,\ldots,u_{n-k+1})}\right]' =$$

$$\frac{W_{n-k}(u_1,\ldots,u_{n-k})W_{n-k+2}(u_1,u_2,\ldots,u_{n-k+1},f)}{(W_{n-k+1}(u_1,\ldots,u_{n-k+1}))^2} .$$

Now it is assumed that

$$\lim_{t \to a^+} \frac{W_{n-k+2}(u_1,\ldots,u_{n-k+1},f)}{W_{n-k+2}(u_1,\ldots,u_{n-k+2})}$$

exists in the induction hypothesis. This means that since $W_{n-k+2}(u_1,\ldots,u_{n-k+2})$ does not change sign in a neighborhood of a, $W_{n-k+2}(u_1,\ldots,u_{n-k+1},f)$ also does not change sign on some neighborhood of a since $[W_{n-k+2}(u_1,\ldots,u_{n-k+1},f)/W_{n-k+2}(u_1,\ldots,u_{n-k+2})]$ is itself monotonic on some neighborhood of a. Therefore $[W_{n-k+1}(u_1,\ldots,u_{n-k},f)/W_{n-k+1}(u_1,\ldots,u_{n-k+1})]$ has a derivative that does not change sign in a neighborhood of a and so it is monotonic and the required limit exists.

The proof at b is exactly the same.

This lemma has a useful corollary.

COROLLARY 2.4 *If* L *is a disconjugate* n^{th} *order linear differential operator on* $[a,b]$ *and if* $Lf \geq 0$ *on* (a,b) *and* $u_1,u_2,\ldots,u_n$ *is a Decartes fundamental principal system of solutions of* L *on* (a,b) *then* $D^{(j)}f(t)$ *exists for* $j = 0,1,\ldots,n-1$ *and* $t = a, b$.

Proof. Since $u_1,u_2,\ldots,u_n$ is Descartes on (a,b) the Wronskians

$W_k(u_1,\ldots,u_k)$ and $W_{k+1}(u_n,\ldots,u_{n-k})$ satisfy the requirements of the last lemma. Also Lf may be written as $[W_{n+1}(u_1,u_2,\ldots,u_n,f)/W_n(u_1,\ldots,u_n)]$ and since $W_n(u_1,\ldots,u_n)$ is positive on (a,b) and $Lf \geq 0$ on (a,b) it follows that $W_{n+1}(u_1,u_2,\ldots,u_n,f)$ doe not change sign on (a,b). Therefore the limits of the lemma exist at a and b and these limits are just the generalized derivatives at a and b.

<u>LEMMA 2.5</u> *Let* $u_1,u_2,\ldots,u_n$ *be a Descartes fundamental principal system of solutions on* (a,b) . *If* $z_j = (u_{j+1}/u_1)'$ *and* $y_j = (-u_j/u_n)'$ *for* $j = 1,2,\ldots,n-1$ *then* $z_1,z_2,\ldots,z_{n-1}$ *and* $y_1,y_2,\ldots,y_{n-1}$ *are Descartes fundamental principal systems of solutions on* (a,b).

<u>Proof</u>. Theorem 1.13 gives that for $i_1 < \ldots < i_k$, $j_1 < \ldots < j_k$

$$\lim_{t \to a(b)} \frac{W(u_{i_1},\ldots,u_{i_k})}{W(u_{j_1},\ldots,u_{j_k})} = 0\,(\infty)$$

if $i_r \geq j_r$, $r = 1,2,\ldots,k$ and strict inequality holds for at least one r. Since $z_k = W_2(u_1,u_{k+1})/u_1^2$, it follows that $\lim_{t\to a^+} (z_{k+1}/z_k) =$ $\lim_{t \to a^+} [W_2(u_1,u_{k+2}/W_2(u_1,u_{k+1})] = 0$ and

$$\lim_{t \to b^-} (z_k/z_{k+1}) = \lim_{t \to b^-} \frac{W_2(u_1,u_{k+1})}{W_2(u_1,u_{k+2})} = 0 .$$

By hypothesis $W_2(u_1,u_{k+1}) > 0$ on (a,b) and so z_k is strictly positive on every neighborhood of a and b. Therefore $z_1,z_2,\ldots,z_{n-1}$ is a fundamental principal system on (a,b). The proof for $y_1,y_2,\ldots,y_{n-1}$ is the same.

To show that these systems are Descartes on (a,b), from Corollary 1.7

$$u_1^{k+1}\, W_k\left(\left(\frac{u_{i_1+1}}{u_1}\right)', \left(\frac{u_{i_2+1}}{u_1}\right)', \ldots, \left(\frac{u_{i_k+1}}{u_1}\right)'\right)$$

$$= W_{k+1}(u_1, u_{i_1+1}, \ldots, u_{i_{k+1}}) > 0$$

for $1 \leq i_1 < i_2 < \ldots < i_k \leq n-1$ and so $W_k(z_{i_1}, z_{i_2}, \ldots, z_{i_k}) > 0$ and $z_1, z_2, \ldots, z_{n-1}$ is a Descartes system on (a,b). For $y_1, y_2, \ldots, y_{n-1}$,

$$W_k\left(\left(-\frac{u_{i_1}}{u_n}\right)', \left(-\frac{u_{i_2}}{u_n}\right)', \ldots, \left(-\frac{u_{i_k}}{u_n}\right)'\right)$$

$$= \left(\frac{1}{u_n}\right)^{k+1} W_{k+1}(u_n, -u_{i_1}, -u_{i_2}, \ldots, -u_{i_k})$$

$$= \left(\frac{1}{u_n}\right)^{k+1} W_{k+1}(u_{i_1}, u_{i_2}, \ldots, u_{i_k}, u_n) > 0$$

and so for $1 \leq i_1 < i_2 < \ldots < i_k \leq n-1$, $W_k(y_{i_1}, y_{i_2}, \ldots, y_{i_k}) > 0$ and $y_1, y_2, \ldots, y_{n-1}$ is therefore Descartes on (a,b).

<u>LEMMA 2.6</u> *If* $S = \{u_1, u_2, \ldots, u_n\}$ *is a Descartes fundamental principal system of solutions on* (a,b) *and* $S_1 = z_1, z_2, \ldots, z_{n-1}$ *and* $S_2 = y_1, y_2, \ldots, y_{n-1}$ *where* z_k *and* y_k *are as in Lemma 2.5 then:*

$$S\, \mathcal{D}^{(j)} u_k(a) = S_1\, \mathcal{D}^{(j-1)} \left(\frac{u_k}{u_1}\right)'(a); \qquad j = 1,2,\ldots,n-1$$

$$S\, \mathcal{D}^{(j)} u_k(b) = S_1\, \mathcal{D}^{(j)} \left(\frac{u_k}{u_1}\right)'(b); \qquad j = 0,1,\ldots,n-2$$

$$S\, \mathcal{D}^{(j)} u_k(a) = S_2\, \mathcal{D}^{(j)} \left(-\frac{u_k}{u_n}\right)'(a); \qquad j = 0,1,\ldots,n-2$$

$$S\, \mathcal{D}^{(j)} u_k(b) = S_2\, \mathcal{D}^{(j-1)} \left(-\frac{u_k}{u_n}\right)'(b); \qquad j = 1,2,\ldots,n-1$$

and if f *is an* n *times differentiable function on* (a,b) *with* $W_{n+1}(u_1,u_2,\ldots,u_n,f) \geq 0$ *on* (a,b) *then these four equalities are true with* u_k *replaced by* f.

Proof. In the case of u_k these generalized derivatives take a particularly simple form for

$$S\,\mathcal{D}^{(j)}u_k(a) = \lim_{t \to a^+} \frac{W_{j+1}(u_1,\ldots,u_j,u_k)}{W_{j+1}(u_1,\ldots,u_{j+1})} = \begin{cases} 1 & \text{if } k = j+1 \\ 0 & \text{if } k \neq j+1 \end{cases}$$

This is because if $k < j+1$, $W_{j+1}(u_1,\ldots,u_j,u_k) \equiv 0$ on (a,b) and if $k > j+1$, Theorem 1.13 gives that

$$\lim_{t \to a^+} \frac{W_{j+1}(u_1,\ldots,u_j,u_k)}{W_{j+1}(u_1,\ldots,u_{j+1})} = 0 \quad .$$

In the same way one has that $S_1\,\mathcal{D}^{(j)}z_k(a) = 1$ if $k = j+1$ and zero otherwise. Therefore $S_1\,\mathcal{D}^{(j-1)}z_{k-1}(a) = S\,\mathcal{D}^{(j)}u_k(a)$ and so $S_1\,\mathcal{D}^{(j-1)}(u_k/u_1)'(a) = S\,\mathcal{D}^{(j)}u_k(a)$.

The derivative $S\,\mathcal{D}^{(j)}u_k(b)$ is

$$\lim_{t \to b^-} \frac{W_{j+1}(u_n,\ldots,u_{n-j+1},u_k)}{W_{j+1}(u_n,\ldots,u_{n-j})} \quad .$$

This is zero if $k \geq n-j+1$ since in that case $W_{j+1}(u_n,\ldots,u_{n-j+1},u_k) \equiv 0$ on (a,b) and if $k < n-j$ Theorem 1.13 gives

$$\lim_{t \to b^-} \frac{W_{j+1}(u_n,\ldots,u_{n-j+1},u_k)}{W_{j+1}(u_n,\ldots,u_{n-j})} = 0 \quad .$$

In a similar way one has that $S_1\,\mathcal{D}^{(j)}z_k(b) = 1$ if $k = (n-1)-j$ and 0

otherwise. This gives therefore that $S_1\, \mathcal{D}^{(j)} z_{k-1}(b) = S\, \mathcal{D}^{(j)} u_k(b)$ or that $S_1\, \mathcal{D}^{(j)}(u_k/u_1)'(b) = S\, \mathcal{D}^{(j)} u_k(b)$.

The proof for the system S_2 is essentially the same.

For $S\, \mathcal{D}^{(j)} f$ one has

$$S\, \mathcal{D}^{(j)} f(a) = \lim_{t \to a^+} \frac{W_{j+1}(u_1,\ldots,u_j,f)}{W_{j+1}(u_1,\ldots,u_{j+1})} .$$

Since on (a,b) it is assumed that $W_{n+1}(u_1,u_2,\ldots,u_n,f) \geq 0$, Corollary 2.4 implies that this limit exists. Corollary 1.7 implies that $W_n(z_1,\ldots,z_{n-1},(f/u_1)') \geq 0$ and so again Corollary 2.4 implies that $S_1\, \mathcal{D}^{(j)}(f/u_1)'$ exists. However

$$S_1\, \mathcal{D}^{(j)}\left(\frac{f}{u_1}\right)'(a) = \lim_{t \to a^+} \frac{W_{j+1}((u_2/u_1)',\ldots,(u_{j+1}/u_1)',(f/u_1)')}{W_{j+1}((u_2/u_1)',\ldots,(u_{j+2}/u_1)')}$$

$$= \lim_{t \to a^+} \frac{W_{j+2}(u_1,u_2,\ldots,u_{j+1},f)}{W_{j+2}(u_1,u_2,\ldots,u_{j+2})} = S\, \mathcal{D}^{(j+1)} f(a)$$

so that $S\, \mathcal{D}^{(j)} f(a) = S_1\, \mathcal{D}^{(j-1)}(f/u_1)'a$. At b, $S\, \mathcal{D}^{(j)} f(b)$ is

$$\lim_{t \to b^-} \frac{W_{j+1}(u_n,\ldots,u_{n-j+1},f)}{W_{j+1}(u_n,\ldots,u_{n-j})}$$

and again this limit exists by Corollary 2.4 since $W_{n+1}(u_1,\ldots,u_n,f) \geq 0$ on (a,b).

Now

$$\lim_{t \to b^-} \frac{W_{j+1}(u_n,\ldots,u_{n-j+1},f)}{W_{j+1}(u_n,\ldots,u_{n-j})}$$

$$= \lim_{t \to b^-} \left[\frac{W_{j+1}(u_n,\ldots,u_{n-j+1},f)}{W_{j+1}(u_n,\ldots,u_{n-j+1},u_1)} \right] \Big/ \left[\frac{W_{j+1}(u_n,\ldots,u_{n-j+1},u_{n-j})}{W_{j+1}(u_n,\ldots,u_{n-j+1},u_1)} \right]$$

and since

$$\lim_{t \to b^-} \frac{W_{j+1}(u_n,\ldots,u_{n-j+1},u_{n-j})}{W_{j+1}(u_n,\ldots,u_{n-j+1},u_1)} = \infty$$

there is, because of Lemma 1.17 a sequence $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{t \to \infty} t_k = b$ and

$$S\,\mathcal{D}^{(j)}f(b) = \lim_{k \to \infty}$$

$$\left[\frac{W_{j+1}(u_n,\ldots,u_{n-j+1},f)}{W_{j+1}(u_n,\ldots,u_{n-j+1},u_1)}\right]'(t_k) \Bigg/ \left[\frac{W_{j+1}(u_n,\ldots,u_{n-j+1},u_{n-j})}{W_{j+1}(u_n,\ldots,u_{n-j+1},u_1)}\right]'(t_k) \quad .$$

Making use of the differentiation formula of Lemma 1.8 gives

$$S\,\mathcal{D}^{(j)}f(b) = \lim_{k \to \infty}$$

$$\frac{[W_j(u_n,\ldots,u_{n-j+1})W_{j+2}(u_n,\ldots,u_{n-j+1},u_1,f)/(W_{j+1}(u_n,\ldots,u_{n-j+1},u_1))^2](t_k)}{[W_j(u_n,\ldots,u_{n-j+1})W_{j+2}(u_n,\ldots,u_{n-j+1},u_1,u_{n-j})/(W_{j+1}(u_n,\ldots,u_{n-j+1},u_1))^2](t_k)}$$

$$= \lim_{k \to \infty} \left| \frac{W_{j+2}(u_n,\ldots,u_{n-j+1},u_1,f)}{W_{j+2}(u_n,\ldots,u_{n-j+1},u_1,u_{n-j})} \right| (t_k)$$

$$= \lim_{k \to \infty} \frac{W_{j+1}((u_n/u_1)',\ldots,(u_{n-j+1}/u_1)',(f/u_1)')\ (t_k)}{W_{j+1}((u_n/u_1)',\ldots,(u_{n-j+1}/u_1)',(u_{n-j}/u_1)')(t_k)}$$

$$= S_1\,\mathcal{D}^{(j)}(f/u_1)'(b),$$

since by Corollary 2.4 this last derivative exists. Therefore $S\,\mathcal{D}^{(j)}f(b) = S_1\,\mathcal{D}^{(j)}(f/u_1)'(b)$.

The proof for $S_2\,\mathcal{D}^{(j)}(-\,f/u_n)$ is similar.

<u>Pólya's Theorem With Singular Endpoints</u>

Having these facts about generalized derivatives one can prove the analogue of Lemma 2.1 in the case of singular endpoints.

<u>LEMMA 2.7</u> *If* $u_1, u_2, \ldots, u_n$ *is a Descartes fundamental principal system on* (a,b) *and* $a \le t_1 < t_2 < \ldots < t_m \le b$ *and* $r_1, r_2, \ldots, r_m$ *are positive integers whose sum is* $n+1$; *then*

i) if $t_1 \neq a$ *and* $t_m = b$ *with* $r_m = n$ *then*

$$W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1, & b \\ 1, & n \end{bmatrix} = u_n(t_1)(-1)(-1)^{n+2}(-1)^{n-1}$$

$$\int_{t_1}^{b} W_n((-u_1/u_n)',\ldots,(-u_{n-1}/u_n)',(f/u_n)')\begin{bmatrix} s, & b \\ 1, & (n-1) \end{bmatrix} ds$$

ii) if $t_1 = a$ *and* $t_m = b$ *and* $r_m \neq n$

$$W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} a, & t_2,\ldots,t_{m-1}, & b \\ r_1, & r_2,\ldots,r_{m-1}, & r_m \end{bmatrix} =$$

$$\prod_{i=2}^{m-1} u_1(t_i)^{r_1} \int_a^{t_2} \int_{t_2}^{t_3} \cdots \int_{t_{m-2}}^{t_{m-1}} W_m((u_2/u_1)',\ldots,(u_n/u_1)',(f/u_1)')$$

$$\begin{bmatrix} a, & s_1, & t_2, & \ldots, & t_{m-1}, & b \\ r_1-1, & 1, & r_2-1, & \ldots, & r_{m-1}-1, & r_m \end{bmatrix} ds_{m-2}\, ds_{m-3} \ldots ds_1$$

iii) if $t_1 \neq a$ *and* $t_m = b$ *but* $r_m \neq n$ *then*

$$W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1, & \ldots, & b \\ r_1, & \ldots, & r_m \end{bmatrix}$$

$$= \prod_{i=1}^{m-1} u_1(t_i)^{r_i} \int_{t_1}^{t_2} \int_{t_2}^{t_3} \cdots \int_{t_{m-2}}^{t_{m-1}} W_n((u_2/u_1)',\ldots,(u_n/u_1)',(f/u_1)')$$

$$\begin{bmatrix} t_1, & s_1, & t_2, & \ldots, s_{m-2}, t_{m-1}, & b \\ r_1-1, & 1, & r_2-1, & \ldots, 1, \quad r_{m-1}-1, & r_m \end{bmatrix} ds_{m-2} \ldots ds_1$$

iv) if $t_1 = a$ *and* $t_m \neq b$

$$W_n+1(u_1,\ldots,u_n,f)\begin{bmatrix} a, & t_2, & \ldots, & t_m \\ r_1, & r_2, & \ldots, & r_m \end{bmatrix} = \prod_{i=2}^{m} u_1(t_i)^{r_i}$$

$$\int_a^{t_2}\int_{t_2}^{t_3} \int_{t_{m-1}}^{t_m} W_n((u_2/u_1)',\ldots,(u_n/u_1)',(f/u_1)')$$

$$\begin{bmatrix} a & s_1 & \ldots & t_{m-1} & s_{m-1} & t_m \\ r_1-1 & 1 & \ldots & r_{m-1}-1 & 1 & r_m-1 \end{bmatrix} ds_{m-1} \ldots ds_1 \quad .$$

Proof. (i) The integral $\int_{t_1}^{b} W_n((-u_1/u_n)',\ldots,(-u_{n-1}/u_n)',(f/u_n)')$ $\begin{bmatrix} s & b \\ 1 & n-1 \end{bmatrix} ds$ is

$$\begin{vmatrix} \int_{t_1}^{b} (-\frac{u_1}{u_n})' & 0 & \cdots & (-1)^{n-2} \\ \vdots & \vdots & & \vdots \\ \int_{t_1}^{b} (-\frac{u_{n-1}}{u_n})' & (-1)^0 & \cdots & 0 \\ \int_{t_1}^{b} (\frac{f}{u_n})' & S_2\mathcal{D}^{(0)}(\frac{f}{u_n})'b & \cdots & (-1)^{n-2}S_2\mathcal{D}^{(n-2)}(\frac{f}{u_n})'(b) \end{vmatrix}_{n \times n}$$

where S_2 refers to generalized derivatives with respect to $S_2 = \{(-u_1/u_n)', \ldots, (-u_{n-1}/u_n)'\}$. The same argument as in Lemma 2.1 justifies the integration being performed in this way.

This determinant is, after integrating,

$$\begin{vmatrix} \frac{u_1}{u_n}(t_1) & 0 & \cdots & (-1)^{n-2} \\ \vdots & \vdots & & \vdots \\ \frac{u_{n-1}}{u_n}(t_1) & 1 & \cdots & 0 \\ S\mathcal{D}^0 f(b) - \frac{f}{u_n}(t_1) & S_2\mathcal{D}^0(\frac{f}{u_n})'b & \cdots & (-1)^{n-2}S_2\mathcal{D}^{(n-2)}(\frac{f}{u_n})'b \end{vmatrix}_{n \times n}$$

where S refers to generalized derivatives taken with respect to $S = \{u_1, u_2, \ldots, u_n\}$. This determinant can be written as:

$(-1)^{n+2} \times$

$$\begin{vmatrix} \frac{u_1}{u_n}(t_1) & 0 & \cdot & \cdots & (-1)^{n-2} \\ \vdots & \vdots & \vdots & & \\ \frac{u_{n-1}}{u_n}(t_1) & \cdot & 1 & \cdots & \cdot \\ 0 & 1 & 0 & \cdots & 0 \\ S\mathcal{D}^0 f(b) - \frac{f}{u_n}(t_1) & -S\mathcal{D}^0 fb & S_2\mathcal{D}^0(\frac{f}{u_n})'b & \cdots & (-1)^{n-2}S_2\mathcal{D}^{(n-2)}(\frac{f}{u_n})'b \end{vmatrix}_{n+1 \times n+1}$$

by putting in the column $(0,\ldots,0,1,-S\mathcal{D}^0 f(b))^T$ between columns 1 and 2 and row $(0,1,0,\ldots,0)$ between rows r-1 and n. Expansion along the new n^{th} row gives the correct determinant.

Next by adding column 2 to column 1 in the new matrix one obtains the determinant

$$(-1)^{n+2} \times$$

$$\begin{vmatrix} \frac{u_1}{u_n}(t_1) & 0 & 0 & \cdots & (-1)^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{u_{n-1}}{u_n}(t_1) & 0 & 1 & & \cdot \\ 1 & 1 & 0 & \cdots & \cdot \\ (-\frac{f}{u_n})(t_1) & -S\mathcal{D}^0 f(b) & -S\mathcal{D}^{(1)} f(b) & \cdots & (-1)(-1)^{n-2} S\mathcal{D}^{(n-1)} f(b) \end{vmatrix}_{n+1 \times n+1}$$

where the last Lemma 2.6 has been used to replace $S_2\mathcal{D}^{(j-1)}(\frac{f}{u_n})'b$ by $-S\mathcal{D}^{(j)} f(b)$.

Multiplying the last row by (-1) and the first column by $u_n(t_1)$ gives:

$$\frac{(-1)(-1)^{n+2}}{u_n(t_1)} \times$$

$$\begin{vmatrix} u_1(t_1) & 0 & 0 & \cdots & (-1)^{n-2} \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & 0 & & \\ \cdot & \cdot & 1 & & \\ u_n(t_1) & 1 & 0 & \cdots & \\ f(t_1) & S\mathcal{D}^0 f(b) & S\mathcal{D}^{(1)} f(b) & \cdots & (-1)^{n-2}\mathcal{D}^{(n-1)} f(b) \end{vmatrix}_{n+1 \times n+1}$$

Now multiplying columns 3 through n+1 by -1 gives

$$\frac{(-1)(-1)^{n+2}(-1)^{n-1}}{u_n(t_1)} \times$$

$$\begin{vmatrix} u_1(t_1) & 0 & 0 & \cdots & (-1)^{n-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & -1 & & \cdot \\ u_n(t_1) & 1 & 0 & & \cdot \\ f(t_1) & S\mathcal{D}^0 f(b) & -S\mathcal{D}^{(1)} f(b) & \cdots & (-1)^{n-1} S\mathcal{D}^{(n-1)} f(b) \end{vmatrix}_{n+1 \times n+1}$$

which is just $W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1, & b \\ 1, & n \end{bmatrix} / u_n(t_1)$, and so number i) is proved.

ii) The proof of this is exactly the same as in the case of Lemma 2.1 except for the endpoints $a = t_1$ and $b = t_2$. At a the fact that $S\mathcal{D}^{(j)} f(a) = S_1 \mathcal{D}^{(j-1)} (f/u_1)'(a)$ is used instead of Leibnitz's rule.

One has

$$W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} a, & \ldots, & b \\ r_1, & \ldots, & r_m \end{bmatrix} =$$

$$\begin{vmatrix}
1 & 0 & \cdots\ 0 & u_1(t_2)\ \cdots & 0\ \cdots & 0 \\
0 & 1 & \cdot & & \cdot & \cdot \\
\cdot & 0 & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & 0 & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & 1 & \cdot & \cdot & (-1)^{r_m-1} \\
\cdot & \cdot & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & 0 & \cdot \\
0 & 0 & 0 & u_n(t_2)\ \cdots & 1 & 0 \\
S\mathcal{D}^0 f(a) & S\mathcal{D}^{(1)} f(a) & \cdots\ S\mathcal{D}^{(r_1-1)} f(a) & f(t_2)\ \cdots & S\mathcal{D}^0 f(b)\ \cdots & (-1)^{r_m-1} S\mathcal{D}^{(r_m-1)} f(b)
\end{vmatrix}_{n+1 \times n+1}$$

This determinant is

$$\prod_{i=2}^{m-1} u_1(t_i)^{r_i}$$

$$\left| \begin{array}{lllll}
1 & 0 & \cdots & 0 & 0 \\
S\mathcal{D}^0 u_2(a) & 1 & & \cdot & \frac{u_2}{u_1}(t_2) - S\mathcal{D}^0 u_2(a) \\
\cdot & \cdot & & \cdot & \\
\cdot & \cdot & & \cdot & \\
\cdot & \cdot & & \cdot & \\
S\mathcal{D}^0 u_n(a) & \cdot & & \cdot & \\
S\mathcal{D}^0 f(a) & S_1\mathcal{D}^0 (\frac{f}{u_1})'a & \cdots & S_1\mathcal{D}^{(r_1-2)}(\frac{f}{u_1})'a & \frac{f}{u_1}(t_2) - S\mathcal{D}^0 f(a)
\end{array}\right.$$

$$\left. \begin{array}{lllll}
0 & \cdots & 0 & \cdots & 0 \\
(\frac{u_2}{u_1})'(t_2) & \cdots & \cdot & & (-1)^{r_m-1} \\
\cdot & & \cdot & & \\
\cdot & & \cdot & & \\
\cdot & & 1 & & \\
(\frac{f}{u_1})'(t_2) & \cdots & S_1\mathcal{D}^0(\frac{f}{u_1})'b & \cdots & (-1)^{r_m-1} S_1\mathcal{D}^{(r_m-1)}(\frac{f}{u_1})'(b)
\end{array}\right|_{n+1 \times n+1}$$

where Lemma 2.6 has been used. Expansion of this determinant along the first row gives the same thing as one gets by working out the integral in ii). The condition that $r_m \neq n$ is required so that $(-1)^{r_m-1}$ which appears in the last column does not appear in the first row.

The proof of parts iii) and iv) are exactly as the proof of Lemma 2.1 and part ii).

This completes the proof of this lemma.

THEOREM 2.8 *Let* $u_1,u_2,\dots,u_n$ *be a Descartes fundamental principal system of solutions on* (a,b). *Let* $f(t)$ *be an* n *times differentiable function on* (a,b). *Let* $a \le t_1 < t_2 < \dots < t_m \le b$ *be* m *points in* $[a,b]$ *and* $r_1,r_2,\dots,r_m$ *be* m *positive integers whose sum is* $n+1$ *Assume that if* $a = t_1$ *then* $|\mathcal{D}^{(j)}f(a)| < \infty$ *for* $j = 0,1,\dots,r_1-1$ *and if* $b = t_m$, $|\mathcal{D}^{(j)}f(b)| < \infty$ *for* $j = 0,1,\dots,r_m-1$. *With these assumptions if* $W_{n+1}(u_1,\dots,u_n,f) \ge 0$ *on* (t_1,t_m) *then* $W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} t_1, & \dots, & t_m \\ r_1, & \dots, & r_m \end{bmatrix} \ge 0$ *and if also* $W_{n+1}(u_1,\dots,u_n,f)(t) > 0$ *for some* $t \in (t_1,t_m)$ *then strict inequality holds in the conclusion.*

Proof. The proof is an induction on the number n.

For $n = 1$ the Descartes fundamental principal system is u_1. The hypothesis gives that $u_1 > 0$ on (a,b) and $W_2(u_1,f) \ge 0$ on (t_1,t_2). These hypothesis imply that $(f/u_1)'$ exists and is non-negative on (t_1,t_2). It must be shown that $W_2(u_1,f)\begin{bmatrix} t_1, & t_2 \\ 1, & 1 \end{bmatrix} \ge 0$. Depending on whether $t_1 = a$ or $t_2 = b$ this determinant can take on various forms, however, since $u_1(t_1)$ and $u_1(t_2)$ are positive if $t_1 \ne a$ and $t_2 \ne b$ it is true that $W_2(u_1,f)\begin{bmatrix} t_1, & t_2 \\ 1, & 1 \end{bmatrix} \ge 0$ if $\begin{vmatrix} 1 & 1 \\ \ell_2 & \ell_2 \end{vmatrix} \ge 0$ where $\ell_1 = \mathcal{D}^0 f(a)$ of $t_1 = a$, $\ell_1 = f(t_1)/u_1(t_1)$ if $t_1 \ne a$. $\ell_2 = \mathcal{D}^0 f(b)$ if $t_2 = b$ and $\ell_2 = f(t_3)/u_1(t_2)$ if $t_2 \ne b$. However since (f/u_1) is nondecreasing in (t_1,t_2), $(f/u_1)(t_2) - (f/u_1)(t_1) \ge 0$ and this is precisely the condition that $\begin{vmatrix} 1 & 1 \\ \ell_1 & \ell_2 \end{vmatrix} \ge 0$, since

$$\mathcal{D}^{(0)}f(a) = \lim_{t \to a^+} (f/u_1)(t) \quad \text{and} \quad \mathcal{D}^{(0)}f(b) = \lim_{t \to b^-} (f/u_1)(t).$$

Suppose now the theorem is true for all Descartes fundamental principal systems of order less than n. The proof divides up into three cases.

Case I. Let $t_m = b$ and let $r_m = n$. Suppose first that $t_1 = a$. The n^{th} row of the matrix whose determinant is $W_{n+1}(u_1,\ldots,u_n,f)\,[{a,\ b \atop 1,\ n}]$ is $(0,1,0,\ldots,0)$. Expanding along this row gives $W_{n+1}(u_1,u_2,\ldots,u_n,f)[{a,\ b \atop 1,\ n}] = (-1)^{n+2}(-1)(-1)^{n-1}$, $W_n((-\ u_1/u_n)',\ldots,(-\ u_{n-1}/u_n)',(f/u_n)')[{a,\ \ b \atop 1,\ n-1}]$. The factor $(-1)^{n+2}$ arises because 1 in the n^{th} row occurs in the second column. The (-1) arises from multiplying the $n+1^{st}$ row of the original matrix to allow for the fact that

$$S\mathcal{D}^{(j)}f(a) = -\ S_2\mathcal{D}^{(j)}(f/u_n)'(a)$$

and

$$S\mathcal{D}^{(j)}f(b) = -\ S_2\mathcal{D}^{(j-1)}(f/u_n)'(b)$$

where

$$S_2 = \{(-\ u_1/u_n)',\ldots,(-\ u_{n-1}/u_n)'\}\ .$$

It is these relations from Lemma 2.6 that permit the rewriting of the determinant. Lastly the factor $(-1)^{n-1}$ arises from multiplying the last $n-1$ columns by -1 in order to fulfill the definition of

$$W_n((-\ u_1/u_n)',\ldots,(-\ u_{n-1}/u_n)',(f/u_n)')[{a,\ b \atop 1,\ n-1}]$$

in regard to the entires

$$(-1)^k\ S_2\mathcal{D}^{(k)}(-\ u_j/u_n)'b$$

and

$$(-1)^k \; S_2 \mathcal{D}^{(k)} (f/u_n)'b \quad , \qquad k = 0,1,\ldots,n-1$$

and $j = 1,\ldots,n-1$.

Now since, by Corollary 1.7,

$$W_n((-\,u_1/u_n)', \ldots, (-\,u_{n-1}/u_n)', (f/u_n)')$$

$$= (\frac{1}{u_n})^{n+1} \; W_{n+1}(u_n,-u_1,\ldots,-u_{n-1},f)$$

$$= (\frac{1}{u_n})^{n+1} \; W_{n+1}(u_1,u_2,\ldots,u_n,f) \geq 0$$

and the induction hypothesis applies to this case.

Next suppose $t_1 \neq a$. Then

$$W_{n+1}(u_1,\ldots,u_n,f)\left[\begin{matrix} t_1, & b \\ 1, & n \end{matrix}\right] = u_n(t_1)(-1)(-1)^{n+2}(-1)^{n-1}$$

$$\int_{t_1}^{b} W_n((-\,u_1/u_n)',\ldots,(-\,u_{n-1}/u_n)',(f/u_n)')\left[\begin{matrix} s, & b \\ 1, & n-1 \end{matrix}\right] ds$$

from the last Lemma 2.7 part i). Since again Corollary 1.7 implies that

$$w_n((-\,u_1/u_n)', \ldots, (-\,u_{n-1}/u_n)', (f/u_n)') \geq 0$$

on (t_1,b) and Lemma 2.5 gives that $\{(-\,u_1/u_n)', \ldots, (-\,u_{n-1}/u_n)'\}$ is a Descartes fundamental principal system on (a,b) the induction hypothesis applies to the integrand and so the integral is non-negative. But further if $W_{n+1}(u_1,\ldots,u_n,f)(t) > 0$ for some t in (t_1,b) then $W_n((-\,u_1/u_n)', \ldots, (-\,u_{n-1}/u_n)',(f/u_n)')(t) > 0$ and on a set of positive measure the induction hypothesis gives that the integrand is strictly positive and thus so is the integral.

<u>Case II</u>. Let $t_m = b$ and let $r_m < n$. If $t_1 = a$ and $r_1 + r_m = n+1$, then expansion along the first row gives

$$W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} a, & b \\ r_1, & r_m \end{bmatrix} = W_n((u_2/u_1)',\dots,(u_n/u_1)',(f/u_1)')\begin{bmatrix} a, & b \\ r_1-1, & r_m \end{bmatrix}$$

and so this eventually reduces to Case I. If $t_1 \neq a$ and $r_1 + r_m = n+1$ then

$$W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} t_1, & b \\ r_1, & r_m \end{bmatrix} = (u_1(t_1))^{r_1}$$

$$W_n((u_2/u_1)',\dots,(u_n/u_1)',(f/u_1)')\begin{bmatrix} t_1 & b \\ r_1-1 & r_m \end{bmatrix}$$

and this also reduces to Case I. If $r_1 + r_m \neq n+1$ then if $t_1 = a$ Lemma 2.7 gives

$$W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} a, & \dots, t_m \\ r_1, & \dots, r_m \end{bmatrix} =$$

$$\prod_{i=2}^{m-1} (u_1(t_i))^{r_i} \int_a^{t_2}\int_{t_2}^{t_3} \cdots \int_{t_{m-2}}^{t_{m-1}} W_n((u_2/u_1)',\dots,(u_n/u_1)'(f/u_1)')$$

$$\begin{bmatrix} a, & s_1, & t_2, & s_2, & \dots, & s_{m-2}, & t_{m-1}, & t_m \\ r_1-1, & 1, & r_2-1, & 1, & \dots, & 1, & r_{m-1}-1, & r_m \end{bmatrix} ds_{m-2} \cdots ds_1 .$$

Now either the induction hypothesis or Case I applies to the integrand and so as before the theorem is true here too.

<u>Case III</u>. Let $t_m \neq b$. If $t_1 = a$ then Lemma 2.7 gives that

$$W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} a, & \dots, & t_m \\ r_1, & \dots, & r_m \end{bmatrix} =$$

$$\prod_{i=2}^{m} u(t_i)^{r_i} \int_a^{t_2}\int_{t_2}^{t_3} \cdots \int_{t_{m-1}}^{t_{m-1}} W_n((u_2/u_1)',\dots,(u_n/u_1)',(f/u_1)')$$

$$\begin{bmatrix} a, & s_1, & t_2, & \dots, & s_{m-1}, & t_m \\ r_1-1, & 1, & r_2-1, & \dots, & 1, & r_m-1 \end{bmatrix} ds_{m-1} \cdots ds_1 .$$

If $t_1 \neq a$ this is Theorem 2.2. Therefore either induction hypothesis or Theorem 2.2. applies and the theorem is true here also.

This completes the proof of the theorem.

As in Theorem 2.1 the condition that f be n times differentiable can be replaced by the condition that f be $AC^{n-1}(a,b)$. The change required in this proof is then in the case $n = 1$ and that may be found in Theorem 2.2.

Some Operators from J.S. Muldowney

If $L_n f$ is an n^{th} order linear differential operator disconjugate on $[a,b]$ then by Theorem 1.14 there exists a Descartes fundamental principal system of solutions $u_1,\ldots,u_n$ for L_n on (a,b). Let $r_1,\ldots,r_m$ be non-negative integers whose sum is $n-1$ and let $a = t_1 < t_2 < \ldots < t_m = b$ be elements in $[a,b]$.

The function $H(t) \equiv W_{n-(r_1+r_m)}(u_{r_1+1},u_{r_1+2},\ldots,u_{n-r_m})$ $\begin{bmatrix} t, & t_2, & \ldots, t_{m-1} \\ 1, & r_2, & \ldots, r_{m-1} \end{bmatrix}$ is a solution of $L_n x = 0$. The solution $H(t)$ has r_i zeros at t_i, $i = 1,2,\ldots,m$. Since L_n is disconjugate on $[a,b]$ and H is a solution of $L_n x = 0$ H has no zeros except at the points t_i. The cases $r_1 = 0$ or $r_m = 0$ are not excluded from this discussion.

Suppose now that $r_1,\ldots,r_m$ are non-negative integers whose sum is $k \leq n-1$. The set

$$A_k = \{G(t) : L_n G(t) = 0,\ Z_G(t_i) \geq r_i,\ i = 1,2,..,n\}$$

is a subspace of the set of solutions of $L_n x = 0$.

LEMMA 2.9 *The space* A_k *has dimension* $n-k$.

Proof. If $k = n-1$ then the solution $H(t)$ has r_i zeros at t_i and if any other solution $f(t)$ also has r_i zeros at t_i then $G(t) = F(t)H(s) - F(s)H(t)$, s a fixed element of (a,b), $s \neq t_i$; $i = 1,2,\ldots,n$, is a solution of $L_n x = 0$ with n zeros in $[a,b]$ Since L_n is disconjugate on $[a,b]$, $G(t) \equiv 0$ and so F is a constant multiple of H. Therefore the dimension of A_{n-1} is $n-(n-1) = 1$.

For an induction let dimension $A_k = n-k$ if $k \leq n-1$ and consider A_{k-1} where $\sum_{i=1}^{m} r_i = k-1$. If t_i are all a or b then $\{u_{r_1+1},\ldots,u_{n-r_m}\}$ are a basis for A_{k-1} and so $\dim A_{k-1} = n-(r_m+r_1) = n-(k-1)$ since $r_1+r_m = k-1$. Suppose therefore that $t_2 \in (a,b)$. The set

$$B_k = \{G(t) : L_n G = 0,\ Z_G(t_j) \geq r_j,\ j = 1,3,4,\ldots,m,\ Z_G(t_2) \geq r_2+1\}$$

has dimension $n-k$ by the induction hypothesis. For any $s \in (a,b)$, $s \neq t_j$, $j = 1,2,\ldots,m$ there is a solution $x(t)$ with $Z_x(t_j) = r_j$, $j = 1,2,\ldots,m$ and $Z_x(s) = (n-1)-(k-1) = n-k$ and so $x \notin B_k$ since L_n is disconjugate on $[a,b]$ and x cannot have r_2+1 zeros at t_2 unless it has n zeros on $[a,b]$. The solution $x(t)$ is in A_{k-1}. Therefore since $B_k \subset A_{k-1}$ the dimension of A_{k-1} is at least $n-k+1$. Suppose the dimension of A_{k-1} were greater than $n-k+1$, say $n-k+2$. Then if $v_1,\ldots,v_{n-k+2}$ is a basis for A_{k-1} at least one v_i is not in B_k. Let $v_1(t) \notin B_k$. Therefore $v_1^{(r_2)}(t_2) \neq 0$. Define $w_j(t) = v_j(t)v_1^{(r_2)}(t_2) - v_1(t)v_j^{(r_2)}(t_2)$ for $j = 2,3,\ldots,n-k+2$. The $w_j(t)$ are linearly independent and $w_j \in B_j$ since $w_j^{(r_2)}(t_2) = v_j^{(r_2)}(t_2)v_1^{(r_2)}(t_2) - v_1^{(r_2)}(t_2)v_j^{(r_2)}(t_2) = 0$. This however contradicts that dimension of B_k is $n-k$ since there are $n-k+1$ $w_j(t)$ and so the

dimension A_{k-1} cannot be more than $n-(k-1)$. Since it is at least that large it must be that $\dim A_{k-1} = n-(k-1)$. This completes the proof of the lemma.

This lemma allows for the definition of some new operators from L_n. Let $r_1,\ldots,r_m$ be non-negative integers whose sum is $k \leq n$ and let $a = t_1 < t_2 < \ldots < t_m = b$ be elements of $[a,b]$. Then the set $\{G(t) : L_n G = 0,\ Z_G(t_i) \geq r_i,\ i = 1,2,\ldots,m\}$ has a basis $G_1,G_2,\ldots,G_{n-k}$. Define L_{n-k} by:

$$L_{n-k}y = \frac{W_{n-k+1}(G_1,G_2,\ldots,G_{n-k},y)}{W_{n-k}(G_1,G_2,\ldots,G_{n-k})}$$

$$L_0 y = y\ .$$

These operators appear in J.S. Muldowney's paper [5] in the theorem which will appear here with a different proof. For this proof it is necessary to define an operator which is closely related to L_{n-k}.

Let 0_{n-k} be defined by:

$$0_{n-k}f = W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_n, & n-k+1\end{bmatrix} \Big/ W_n(u_1,\ldots,u_n)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k\end{bmatrix}$$

for $t \in (a,b)$ and $u_1,u_2,\ldots,u_n$ a Descartes fundamental principal system for L_n.

<u>LEMMA 2.10</u> *If* L_n *is a linear differential operator which is disconjugate on* $[a,b]$ *and if* f *is* n *times differentiable on* (a,b) *and* $L_n f \geq 0$ *then* $L_{n-k}(f-F) = 0_{n-k}f$ *where* $F(t)$ *is a solution of* $L_n x = 0$ *on* (a,b) *and* $F^{(j)}(t_i) = f^{(j)}(t_i)$, $j = 0,1,\ldots,r_i-1$, $i = 2,\ldots,m-1$, $\mathcal{D}^{(j)}F(a) = \mathcal{D}^{(j)}f(a)$, $j = 0,1,\ldots,r_1-1$; $\mathcal{D}^{(j)}F(b) = \mathcal{D}^{(j)}f(b)$, $j = 0,1,\ldots,r_m-1$ *and* $t_1,\ldots,t_m$ *are points in* $[a,b]$ *used in*

the definition of L_{n-k} *and* 0_{n-k}.

Proof. The first thing is to show that such a function $F(t)$ exists. That it does exist is a consequence of a theorem of Pólya ([7], Theorem II, page 313). That theorem says there is one and only one solution of the equation $L_n x = 0$ on (a,b) assuming n given values at n given points in (a,b) if L_n has a Markov system of solutions on (a,b). In the case L_n is disconjugate on $[a,b]$ the proof is particularly simple, for since there exists a Descartes fundamental principal system of solutions on (a,b), say $u_1, u_2, \ldots, u_n$; the n values at n points give n equations in n unknowns, $c_1, c_2, \ldots, c_n$, the coefficients of the $u_i(t)$. These equations have one and only one solution since the only solution of the corresponding homogeneous system of equations is the identically zero solution.

In the present case if f has r_1 zeros at a and r_m zeros at b then, if $u_1, u_2, \ldots, u_n$ is a Descartes fundamental principal system of solutions for $L\ y = 0$ on (a,b),

$$\sum_{i = r_1+1}^{n-r_m} c_i u_i(t) = F(t)$$

will have the appropriate zeros at a and b. To determine the c_i, $i = r_1+1, \ldots, n-r_m$, there are $n-(r_1+r_m)$ functions $u_{r_1+1}, \ldots, u_{n-r_m}$ and so the Polya theorem permits $n-(r_1+r_m)$ values to be assigned in (a,b). Here however, one needs $k-(r_1+r_m)$ values to be assigned and since $k-(r_1+r_m) \leq n-(r_1+r_m)$ there is at least one choice for the function $F(t)$.

To prove the lemma itself one has

$$L_{n-k}(f-F) = \frac{W_{n-k+1}(G_1,\ldots,G_{n-k},f-F)}{W_{n-k}(G_1,\ldots,G_{n-k})}$$

and

$$O_{n-k}f = W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k+1 \end{bmatrix} \Big/ W_n(u_1,\ldots,u_n)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k \end{bmatrix}.$$

Confining attention to $W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k+1 \end{bmatrix}$ and using the fact that $F(t) = \sum_{j=1}^{n} d_j u_j(t)$ one may write

$$W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k+1 \end{bmatrix}$$

as

$$W_{n+1}(u_1,\ldots,u_n,f-F)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k+1 \end{bmatrix}$$

by adding multiples of the first n rows to the $n+1^{st}$ row and thus not changing the value of the determinant. Now since $G_1,\ldots,G_{n-k}$ are solutions of $L_n x = 0$; $G_i = \sum_{j=1}^{n} c_{ij}u_j(t)$ and the quotient

$$\frac{W_{n+1}(u_1,\ldots,u_n,f-F)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k+1 \end{bmatrix}}{W_n(u_1,\ldots,u_n)\begin{bmatrix} t_1,\ldots,t_m, & t \\ r_1,\ldots,r_m, & n-k \end{bmatrix}}$$

can be rewritten as

$$\frac{\begin{vmatrix} \mathcal{D}^0 u_{i_1}(a) & \cdots & \mathcal{D}^{(r_1-1)} u_{i_1}(a) & u_{i_1}(t_2) & \cdots & (-1)^{r_m-1}\mathcal{D}^{(r_m-1)} u_{i_1}(b) & u_{i_1}(t) & \cdots & u_{i_1}^{(n-k)}(t) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{D}^0 u_{i_k}(a) & \cdots & \mathcal{D}^{(r_1-1)} u_{i_k}(a) & u_{i_k}(t_2) & \cdots & (-1)^{r_m-1}\mathcal{D}^{(r_m-1)} u_{i_k}(b) & u_{i_k}(t) & \cdots & u_{i_k}^{(n-k)}(t) \\ 0 & \cdots & 0 & 0 & \cdots & 0 & G_1(t) & \cdots & G_1^{(n-k)}(t) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & G_{n-k}(t) & \cdots & G_{n-k}^{(n-k)}(t) \\ 0 & \cdots & 0 & 0 & \cdots & 0 & (f-F)(t) & \cdots & (f-F)^{n-k)}(t) \end{vmatrix}_{n+1\ n+1}}{\begin{vmatrix} \mathcal{D}^0 u_{i_1}(a) & \cdots & \mathcal{D}^{(r_1-1)} u_{i_1}(a) & u_{i_1}(t_2) & \cdots & (-1)^{r_m-1}\mathcal{D}^{(r_m-1)} u_{i_1}(b) & u_{i_1}(t) & \cdots & u_{i_1}^{(n-k-1)}(t) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{D}^0 u_{i_k}(a) & \cdots & \mathcal{D}^{(r_1-1)} u_{i_k}(a) & u_{i_k}(t_2) & \cdots & (-1)^{r_m-1}\mathcal{D}^{(r_m-1)} u_{i_k}(b) & u_{i_k}(t) & \cdots & u_{i_k}^{(n-k-1)}(t) \\ 0 & \cdots & 0 & 0 & \cdots & 0 & G_1(t) & \cdots & G_1^{(n-k-1)}(t) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & G_{n-k}(t) & \cdots & G_{n-k}^{(n-k-1)}(t) \end{vmatrix}_{n\times n}}$$

by adding suitable multiples of rows to other rows and interchanging simultaneously the same pairs of rows in the top and bottom determinant. In forming the G_i multiplication of corresponding rows by a nonzero constant leaves the quotient unchanged. The fact that

$$\frac{W_{k+1}(u_1,\ldots,u_k,g)}{W_k(u_1,\ldots,u_k)} + \frac{c\, W_{k+1}(u_1,\ldots,u_k,h)}{W(u_1,\ldots,u_k)} = \frac{W_{k+1}(u_1,\ldots,u_k,g+ch)}{W_k(u_1,\ldots,u_k)}$$

is also used so that if $G_i = \sum_{j=1}^{n} c_{ij}u_j$ then $\mathcal{D}^{(k)}G_i = \sum_{j=1}^{n} c_{ij}\mathcal{D}^{(k)}u_j$. The large blocks of zeros arise from the fact that $Z_{G_i}(t_j) \geq r_j$ and $f^{(j)}(t_i) = F^{(j)}(t_i)$, $j = 0,1,\ldots,r_i-1$, $i = 1,2,\ldots,m$.

The resulting quotient has the form

$$\left|\begin{array}{c:c} A & B \\ \hdashline 0 & C \end{array}\right| \Big/ \left|\begin{array}{c:c} A & D \\ \hdashline 0 & E \end{array}\right|$$

and A is a $k \times k$ matrix. Therefore

$$0_{n-k},f = \frac{|A|\ \ |C|}{|A|\ \ |E|} = \frac{W_{n-k+1}(G_1,\ldots,G_{n-k},f-F)}{W_{n-k}(G_1,\ldots,G_{n-k})} = L_{n-k}(f-F)$$

and this concludes the proof of the lemma.

The Main Theorem

THEOREM 2.11 *Let* L_n *be an* n^{th} *order linear differential operator defined by* $L_nx = x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x$, $p_i(t) \in C(a,b)$, *which is disconjugate on* $[a,b]$. *Let* $a \leq t_1 < t_2 < \ldots < t_m \leq b$ *be* m *points in* $[a,b]$. *Let* f *be a function which is* n *times differentiable on* (a,b) *such that* $Z_f(t_i) \geq r_i$ *where* $r_1,r_2,\ldots,r_m$

are positive integers such that $\sum_{i=1}^{m} r_i = k \leq n$, *and if* $t_1 = a$

$|\mathcal{D}^{(j)}f(a)| < \infty$, $j = 0,1,\ldots,r_1-1$ *and if* $t_m = b$

$|\mathcal{D}^{(j)}f(b)| < \infty$, $j = 0,1,\ldots,r_m-1$.

If $L_n f \geq 0$ *on* (a,b) *then* $p_k(t)\, L_{n-k}f(t) \geq 0$ *on* (a,b) *and if further on* (t_1,t_m), $L_n f \not\equiv 0$ *then* $p_k(t)\, L_{n-k}f(t) > 0$ *for* $t \in (t_1,t_2) \cup \ldots \cup (t_{m-1},t_m)$. *The function* $p_k(t)$ *is* $\prod_{i=1}^{m}$ *sign* $(t-t_i)^{r_i}$.

Proof. This is a somewhat restricted version of Muldowney's Theorem 2.1 in [5]. In that theorem the conditions on $\mathcal{D}^{(j)}f(a)$ and $\mathcal{D}^{(j)}f(b)$ are:

i) $\bar{\mathcal{D}}^{(j)}(-1)^{n-r_1}f(a) \geq 0$, $j = 0,1,\ldots,r_1-1$

ii) $\bar{\mathcal{D}}^{(j)}(-1)^{r_m}f(b) \geq 0$, $j = 0,1,\ldots,r_m-1$.

Lemma 2.10 gives that $0_{n-k}f = L_{n-k}(f-F)$. Since in this theorem $Z_f(t_j) \geq r_j$ let $F(t) \equiv 0$ and then F satisfies the conditions of Lemma 2.10 and one has $0_{n-k}f = L_{n-k}f$.

Let t be fixed in some interval (t_j,t_{j+1}). Then

$$p_k(t)L_{n-k}f = \frac{p_k(t)\; W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1, & \ldots, & t_m, & t \\ r_1, & \ldots, & r_m, & n-k+1 \end{bmatrix}}{W_n(u_1,\ldots,u_n)\begin{bmatrix} t_1, & \ldots, & t_m, & t \\ r_1, & \ldots, & r_m, & n-k \end{bmatrix}} .$$

By interchinging columns this can be written as

$$\frac{p_k(t)\, W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1,\ldots,t_j, & t, & t_{j+1},\ldots,t_m \\ r_1,\ldots,r_j, & n-k+1, & r_{j+1},\ldots,r_m \end{bmatrix}(-1)^{(n-k+1)(r_m+r_{m-1}+\ldots+r_{j+1})}}{W_n(u_1,\ldots,u_n)\begin{bmatrix} t_1,\ldots,t_j, & t, & t_{j+1},\ldots,t_m \\ r_1,\ldots,r_j, & n-k, & r_{j+1},\ldots,r_m \end{bmatrix}(-1)^{(n-k)(r_m+r_{m-1}+\ldots+r_{j+1})}}$$

$$= \frac{p_k(t)\, W_{n+1}(u_1,\ldots,u_n,f)\begin{bmatrix} t_1,\ldots,t_j, & t, & t_{j+1},\ldots,t_m \\ r_1,\ldots,r_j, & n-k+1, & r_{j+1},\ldots,r_m \end{bmatrix}(-1)^{r_m+r_{m-1}+\ldots+r_{j+1}}}{W_n(u_1,\ldots,u_n)\begin{bmatrix} t_1,\ldots,t_j, & t, & t_{j+1},\ldots,t_m \\ r_1,\ldots,r_j, & n-k, & r_{j+1},\ldots,r_m \end{bmatrix}} .$$

Now $p_k(t) = \prod_{i=1}^{m} \text{sign } (t-t_i)^{r_i} = (-1)^{r_m+r_{m-1}+\dots+r_{j+1}}$ and so

$$p_k(t)L_{n-k}f(t) = \frac{W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} t_1,\dots, & t, & \dots,t_m \\ r_1,\dots, & n-k+1, & \dots,r_m \end{bmatrix}}{W_n(u_1,\dots,u_n)\begin{bmatrix} t_1,\dots, & t, & \dots,t_m \\ r_1,\dots, & n-k, & \dots,r_m \end{bmatrix}}$$

and by Theorem 2.8 since $L_n f \geq 0$ on (a,b), $W_{n+1}(u_1,\dots,u_n,f) \geq 0$ on (a,b) and so

$$W_{n+1}(u_1,\dots,u_n,f)\begin{bmatrix} t_1, & \dots, & t, & \dots, & t_m \\ r_1, & \dots, & n-k+1, & \dots, & r_m \end{bmatrix} \geq 0$$

and if inequality is strict in the first it is also strict in the second. Also $W_n(u_1,\dots,u_n)\begin{bmatrix} t_1, & \dots, & t_m \\ r_1, & \dots, & r_m \end{bmatrix} > 0$ and so $p_k(t)\, L_{n-k}f(t) \geq 0$ and if $L_n f \not\equiv 0$ on (t_1,t_m) then $p_k(t)L_{n-k}f(t) > 0$.

This concludes the proof.

Example

Let $L_2 f = (D^2-1)f$. Then e^{-t}, e^t is a Descartes fundamental principal system of solutions on $(-\infty, \infty)$ for $L_2 y = 0$. Suppose $L_2 f \geq 0$ on $(-\infty, \infty)$.

i) If $\lim_{t \to -\infty} \frac{f(t)}{e^{-t}} = 0$ then $f' - f \geq 0$. The condition on the limit is the condition that $Z_f(-\infty) \geq 1$. Therefore if $t_1 = -\infty$ and $r_1 = 1 = k$ in the last theorem then $L_{n-k}f$ is just $(W_2(e^t,f)/e^t)$ since a basis for $\{G : L_2 G = 0;\ Z_G(-\infty) \geq 1\}$ is e^t. So $L_2 f \geq 0 \Rightarrow p_k(t)L_{n-k}f \geq 0$ becomes $f'' - f \geq 0 \Rightarrow (W(e^t,f)/e^t) \geq 0$ which gives $f' - f \geq 0$.

ii) If $\lim_{t \to \infty} \frac{f(t)}{e^t} = 0$ then $f' + f \leq 0$. The limit condition assures that $Z_f(\infty) \geq 1$ and so $p_k(t)L_{n-k}f = \text{sign}\ (t - \infty)(W(e^{-t}f)/e^{-t}) \geq 0$. This implies that $(-1)(f' + f) \geq 0$ or that $f' + f \leq 0$.

iii) If $f(t_1) = 0$ then $(t-t_1)[f'(t)-f(t)\ \coth\ (t-t_1)] \geq 0$. The condition assumed here is that $Z_f(t_1) \geq 1$ for some $t_1 \in (-\infty, \infty)$. To find L_{n-k} it is necessary to find a basis for $\{G : L_2G = 0; Z_G(t_1) \geq 1\}$. The basis required is $G = e^te^{-t_1} - e^{-t}e^{t_1}$ and so

$$L_{2-1}f(t) = \frac{W_2(e^{t-t_1} - e^{-t+t_1}, f)}{(e^{t-t_1} - e^{-t+t_1})} = f' - f\ \coth\ (t-t_1) \quad .$$

Therefore $p_k(t)\ L_{n-k}f(t) \geq 0$ becomes $\text{sign}\ (t-t_1)[f' - f\ \coth\ (t-t_1)] \geq 0$.

iv) If $\lim_{t \to \infty} \frac{f}{e^t} = \lim_{t \to -\infty} \frac{f}{e^{-t}} = 0$ then $f \leq 0$.

Here $k = 2$ and $t_1 = -\infty$ while $t_2 = \infty$. The limit conditions give $Z_f(t_1) \geq 1$ and $Z_f(t_2) \geq 1$. $L_{n-k}f$ is now simply f and $p_k(t)f = (-1)f \geq 0$.

v) If $f(t_1) = f'(t_1) = 0$ then $(t-t_1)^2f(t) \geq 0$. Here again $k = 2$ but now $t_1 \in (-\infty, \infty)$ and $r_1 = 2$. It is assumed that $Z_f(t_1) \geq 2$ and since $k = 2$ again $L_{n-k}f = f$. Here $p_k(t) = \text{sign}\ (t-t_1)^2$, and so the conclusion of the theorem that $p_k(t)L_{n-k}f(t) \geq 0$ gives $(t-t_1)^2f(t) \geq 0$.

vi) If $\lim_{t \to -\infty} \frac{f}{e^{-t}} = 0$ and $\lim_{t \to -\infty} \frac{f'+f}{e^t} = 0$ then $f' \geq 0$. The first limit from Example i) gives that $f'-f \geq 0$. If one lets $Ly = (D-1)y$ then if $y = f'+f$, $Ly = L_2f \geq 0$. The theorem applied to L gives $y \geq 0$ or $f' + f \geq 0$. This gives, adding $f'+f$ and

$f'-f$ that $2f' \geq 0$ or $f' \geq 0$.

A Corollary: Pólya's Mean Value Theorem

COROLLARY 2.12 *If* $L_n f$ *is disconjugate on* $[a,b]$ *and if* f *is* n *times differentiable on* (a,b) *and has* $n+1$ *zeros on* $[a,b]$ *then there exists a* $c \in (a,b)$ *such that* $L_n f(c) = 0$.

Proof. Let $t_1,\ldots,t_m$ be the zeros of order $r_1,\ldots,r_m$ respectively of f. Then if $L_n f > 0$ on (a,b) the last theorem is satisfied if $r_1,\ldots,r_m$ is replaced by $r_1-1,r_2,\ldots,r_m$ and also $r_1,r_2,\ldots,r_m-1$ and so $p_k(t)f \geq 0$ holds for two different functions $p_k(t)$ of opposite sign so that $f = 0$. The same argument works if it is assumed $L_n f < 0$ with $-f$ replacing f. Thus the Darboux intermediate value property gives that $L_n f(c) = 0$ for some $c \in (a,b)$. This concludes the proof which is taken as it appears on page 88 as Corollary 2.1 in [5].

Example: The case $L_n f = f^{(n)}$ is just repeated application of Rolle's theorem. In the case $n = 1$ the corollary is seen in another way. If f has 2 zeros in $[a,b]$, a, b finite, then $f'(t) + p(t)f(t)$ has a zero on (a,b). Let

$$g(t) = e^{\int_a^t p(t)} f(t)$$

and then g has 2 zeros on $[a,b]$ and

$$g' = e^{\int_a^t p(t)} [f'(t) + p(t)f'(t)]$$

has at least one zero in (a,b) and so $f' + pf$ has a zero in (a,b).

This corollary in the case of nonsingular endpoints was originally due to Pólya in [7].

CHAPTER III

ABSTRACT

The last chapter is an application of the main theorem of Chapter II.

The purpose is to illustrate, by means of special examples, that if f is n times differentiable and L_n is disconjugate on $[a,b]$ then bounds may be obtained for $M_k f$ in terms of bounds on $L_n f$ if M_k is a linear differential operator of order k, $0 \leq k \leq n$. The examples considered pertain to the cases $n = 1,2,3$ and L_n an operator with constant coefficients.

CHAPTER III

APPLICATION

Preliminaries

In [6], J.S. Muldowney discussed the relationships between bounds on differential operators. In that work he used Theorem 2.11 and two of its corollaries to examine the operator given by $L_2 e^{\lambda t} = (\lambda-\ell_1)(\lambda-\ell_2)e^{\lambda t}$ where ℓ_1 and ℓ_2 are real valued functions. Under certain conditions he obtained bounds on f, $f'-cf$ and $bf'-f$ in terms of bounds on c, b and $L_2 f$ where c and b are real functions. In the case that $\ell_1(t)$ and $\ell_2(t)$ are constant he obtained bounds on $f'' + p_1(t)f' + p_2(t)f$ in terms of ℓ_1, ℓ_2, $p_1(t)$, $p_2(t)$ and $L_2 f$, where $p_1(t)$ and $p_2(t)$ are real functions.

In this chapter bounds on f, $M_1 f$, $M_2 f$ and $M_3 f$ are given in terms of operators $L_1 = D-\ell_1$, $L_2 = (D-\ell_1)(D-\ell_2)$ and $L_3 = (D-\ell_1)(D-\ell_2)(D-\ell_3)$ where ℓ_1, ℓ_2, and ℓ_3 are real constants and M_1, M_2, M_3 are general first, second and third order linear differential operators with leading coefficient 1.

To begin it is necessary to have the two corollaries mentioned above. These corollaries are Corollary 1.1 and Corollary 1.2 respectively of [6], where they appear on page 108.

COROLLARY 3.1 *Let* L_n *be an* n^{th} *order linear differential operator which is disconjugate on* $[a,b]$ *and let* $a \leq t_1 < t_2 < \cdots < t_m \leq b$ *be* m *points in* $[a,b]$. *Let* $r_1, r_2, \ldots, r_m$ *be* m *positive integers whose sum is less than or equal to* n. *Suppose* f *is a function* n *times*

differentiable on (a,b) *and* $Z_f(t_i) \geq r_i$; $i = 1,2,\ldots,m$. *If* $L_n f \geq 0$ *on* (a,b) *and* $L_{n-k} f = 0$ *for some* $t \neq t_i$ *then* $L_n f = 0$ *and* $L_{n-k} f = 0$ *on every interval* (t_i,t) *and* (t,t_i); $i = 1,2,\ldots,m$. *Also on the union of those intervals* $f = c_1 G_1 + c_2 G_2 + \ldots + c_{n-k} G_{n-k}$ *where* $\{G_1,\ldots,G_{n-k}\}$ *is a basis for* $\{x : L_n x = 0;\ Z_x(t_i) \geq r_i;\ i = 1,2,\ldots,m\}$.

Proof. If $L_n f \not\equiv 0$ on one of (t_i,t) or (t,t_i) then by Theorem 2.11 $p_k\, L_{n-k} f(t) > 0$ and since $L_{n-k} f(t)$ is assumed to be zero it must be that $L_n f \equiv 0$ on every (t,t_i) and (t_i,t).

If $L_{n-k} f(t) = 0$ then

$$\frac{W_{n-k+1}(G_1,\ldots,G_{n-k},f)(t)}{W_{n-k}(G_1,\ldots,G_{n-k})(t)} = 0$$

implies the column $[f(t),f'(t),\ldots,f^{(n-k)}(t)]^T$ of $W_{n-k+1}(G_1,\ldots,G_{n-k},f)(t)$ is a linear combination of the preceeding columns and so

$$f^{(i)}(t) = c_1 G_1^{(i)}(t)+\ldots+c_{n-k} G_{n-k}^{(i)}(t); \quad i = 1,2,\ldots,n-k$$

for some constants $c_1,c_2,\ldots,c_{n-k}$. However if $F = f-c_1G_1-\ldots-c_{n-k}G_{n-k}$ then $Z_F(t) \geq n-k$ and so if I is the union of the intervals $[t,t_i]$ and $[t_i,t]$ then $Z_F(I) \geq n$. But $L_n F = 0$ on I and so since L_n is disconjugate on I, $F \equiv 0$ on I.

COROLLARY 3.2 *If* f *and* g *are* n *times differentiable functions and* $L_n g > 0$ *and* $Z_f(t_i) \geq r_i$ *and* $Z_g(t_i) \geq r_i$ *then*

$$p_k(t)[L_{n-k} f(t) + L_{n-k} g(t)\ \| L_n f / L_n g \|] \geq 0$$

and

$$p_k(t)[L_{n-k}f(t) - L_{n-k}g(t)||L_nf/L_ng||] \leq 0$$

where L_n, t_i, r_i *are as in Corollary 3.1 and* $||\cdot||$ *is the supremum norm on* $L^\infty[a,b]$.

Further, under these conditions,

$$|L_{n-k}f(t)| \leq |L_{n-k}g(t)| \; ||L_nf/L_ng||$$

and equality holds only if $f = cg + c_1G_1 + \ldots + c_{n-k}G_{n-k}$ *on all intervals* $[t,t_i]$ *and* $[t_i,t]$; $i = 1,2,\ldots,m$.

Proof. Let $F = f+g\,||L_nf/L_ng||$. Then F satisfies the conditions of Theorem 2.11 since $Z_F(t_i) \geq \min\,(Z_f(t_i),\, Z_g(t_i)) \geq r_i$ and $L_nF = L_nf + ||L_nf/L_ng||\; L_ng \geq 0$. Therefore $p_k\, L_{n-k}F \geq 0$ on $[a,b]$ so $p_k(t)[L_{n-k}f(t) + ||L_nf/L_ng||\; L_{n-k}g(t)] \geq 0$.

Let $F = -\,f + g\;||L_nf/L_ng||$ and Theorem 2.11 again gives that $p_k\, L_{n-k}F \geq 0$ or in this case that

$$p_k(t)[L_{n-k}f(t) - L_{n-k}g(t)\;||L_nf/L_ng||] \leq 0\;.$$

Since $L_ng > 0$ Theorem 2.11 implies that $p_k\, L_{n-k}g \geq 0$ and so if $p_k > 0$, $L_{n-k}g \geq 0$ and if $p_k < 0$, $L_{n-k}g \leq 0$. This and the first inequality give

$$L_{n-k}f \geq -\,L_{n-k}g\;||L_nf/L_ng|| \qquad \text{for} \qquad L_{n-k}g \geq 0$$

$$L_{n-k}f \leq -\,L_{n-k}g\;||L_nf/L_ng|| \qquad \text{for} \qquad L_{n-k}g \leq 0\;.$$

The second inequality gives

$$L_{n-k}f \leq L_{n-k}g \; ||L_n f/L_n g|| \qquad \text{for} \quad L_{n-k}g \geq 0$$

$$L_{n-k}f \geq L_{n-k}g \; ||L_n f/L_n g|| \qquad \text{for} \quad L_{n-k}g \leq 0 \quad .$$

Therefore

$$- L_{n-k}g \; ||L_n f/L_n g|| \leq L_{n-k}f \leq L_{n-k}g \; ||L_n f/L_n g||$$

if $L_{n-k}g \geq 0$ and

$$L_{n-k}g \; ||L_n f/L_n f|| \leq L_{n-k}f \leq - L_{n-k}g \; ||L_n f/L_n g||$$

if $L_{n-k}g \leq 0$. These inequalities may be written as

$$|L_{n-k}f| \leq |L_{n-k}g| \; ||L_n f/L_n g|| \; .$$

Equality can occur at t only when $L_{n-k}F(t) = 0$ and that happens only when $F = c_1G_1 + \ldots + c_{n-k}G_{n-k}$ or $f = cg + c_1G_1 + \ldots + c_{n-k}G_{n-k}$.

The next lemma is a remark on page 60 in Levin's paper [4] which is there given without proof.

<u>LEMMA 3.3</u> *If $\ell_1, \ell_2, \ldots, \ell_m$ are distinct real numbers and if $L_m = (D-\ell_1), \ldots, (D-\ell_m)$ and $L_n = (D-\ell_1)^{k_1}, \ldots, (D-\ell_m)^{k_m}$ with $\sum_{i=1}^{m} k_i = n > m$ then L_m is disconjugate on $[-\infty, \infty]$ and L_n is disconjugate on $(-\infty, \infty)$ but not on $[-\infty, \infty]$.*

<u>Proof</u>. Let $\ell_1 < \ell_2 < \ldots < \ell_m$. Then for L_m the set $\{e^{\ell_1 t}, e^{\ell_2 t}, \ldots, e^{\ell_m t}\}$ is a Descartes fundamental principal system of solutions on

$(-\infty, \infty)$ and so by Theorem 1.16, L_m is disconjugate on $[-\infty, \infty]$.

The demonstration that L_n is disconjugate on $(-\infty, \infty)$ is by an induction. For $n = 1$ the solutions of $L_1 x = 0$ are $c\,e^{\ell_1 t}$ where c is a constant. These solutions have no zeros unless they are identically zero. For the induction assume that for such operators the lemma is true for orders less than n, $n > 1$. Suppose for a contradiction that some solution x of $L_n x = 0$ has n zeros in $(-\infty, \infty)$. Then x is a linear combination of $e^{\ell_1 t}$, $te^{\ell_1 t}$, ..., $t^{k_1-1}e^{\ell_2 t}$, $te^{\ell_2 t}$, ..., $t^{k_2-1}e^{\ell_2 t}$, ..., $e^{\ell_m t}$, ..., $t^{k_m-1}e^{\ell_m t}$. Suppose $x = c_1 e^{\ell_1 t} + c_2 te^{\ell_1 t} + \ldots + c_{k_1+\ldots+k_m} t^{k_m-1} e^{\ell_m t}$. Then

$(xe^{-\ell_1 t})' = 2c_2 t + \ldots + (k_1-1)c_{k_1} t^{k_1-2} + d_{k_1+1} e^{(\ell_2-\ell_1)t} + \ldots +$ $d_{k_1+k_2} t^{k_2-1} e^{(\ell_2-\ell_1)t} + \ldots + d_{k_1+\ldots+k_{m-1}+1}\, e^{(\ell_m-\ell_1)t} + \ldots +$ $d_{k_1+\ldots+k_m} t^{k_m-1} e^{(\ell_m-\ell_1)t}$ and is a solution of $L_{n-1}x = 0$ where

$L_{n-1} = \prod_{i=1}^{m} (D-\bar{\ell}_i)^{\bar{k}_i}$ where $\bar{\ell}_i = \ell_i - \ell_1$ and $\bar{k}_i = k_i$, $i = 2,3,\ldots,m$ and $\bar{k}_1 = k_1 - 1$. By the induction hypothesis this solution has at most $n-2$ zeros on $(-\infty, \infty)$. However if x has n zeros on $(-\infty, \infty)$ then $(xe^{-\ell_1 t})'$ has $n-1$ zeros there and this contradiction proves this part of the lemma.

That L_n is not disconjugate on $[-\infty, \infty]$ may be seen by exhibiting a solution with more than $n-1$ zeros on $[-\infty, \infty]$. A principal system at ∞ is $e^{\ell_1 t}, te^{\ell_1 t}, \ldots, t^{k_1-1}e^{\ell_1 t}, e^{\ell_2 t}, \ldots, t^{k_m-1}e^{\ell_m t}$. A principal system of solutions at $-\infty$ is $e^{\ell_m t}, (-t)e^{\ell_m t}, \ldots,$

$e^{\ell_2 t},\ldots,e(-t)^{k_2-1}e^{\ell_2 t},e^{\ell_1 t}, -te^{\ell_1 t}, \ldots, (-t)^{k_1-1}e^{\ell_1 t}$. Using these two principal systems of solutions gives that $e^{\ell_1 t}$ has $n-1$ zeros at ∞ and $n-(n-k+1)$ zeros at $-\infty$. Thus if $k_1 > 1$ $e^{\ell_1 t}$ has at least n zeros on $[-\infty, \infty]$. If $k_1 = 1$ then a similar argument applied to any ℓ_j such that $k_j > 1$ will work.

COROLLARY 3.4 *If* L_n *is as in Lemma 3.3 then* L_n *is disconjugate on* $(-\infty, \infty]$ *and* $[-\infty, \infty)$.

Proof. This is an application of Lemma 1.5 which gives that L_n is disconjugate on the half closed interval if it is disconjugate on the open interval.

To conclude this chapter these preliminaries will be applied to a particular first order operator and then a second order operator and lastly a third order operator all of which operators are linear with constant coefficients.

Inequalities Involving a First Order Operator

PROPOSITION 3.5 *Let* $L_1 = (D-\ell_1)$ *with* $0 < \ell_1$ *a real number and let* f *be a differentiable function on* $(-\infty, \infty)$ *such that* $\lim_{t \to \infty} e^{-\ell_1 t} f(t) = 0$. *Then* $|f| \leq (||L_1 f||/\ell_1)$, *and*

$$|M_1 f| \leq |L_1 f| + (||L_1 f||/\ell_1)\,|\ell_1 + B(t)|$$

where $M_1 f \equiv f' + B(t)f(t)$ *is any first order linear differential operator.*

Proof. The function $e^{\ell_1 t}$ is a fundamental principal system for L_1 on $(-\infty, \infty)$. Let $g = -1$. Then $L_1 g = \ell_1 > 0$. Since $\lim_{t \to \infty} e^{-\ell_1 t} g = 0$, $Z_g(\infty) \geq 1$ and also by the hypothesis $Z_f(\infty) \geq 1$. Applying Corollary 3.2 with $k = 1$, $t_1 = \infty$, $r_1 = 1$ the resulting inequality is

$$|L_o f| \leq |L_o g| \; (||L_1 f||/||L_1 g||)$$

which is $|f| \leq |-1| \; (||L_1 f||/||L_1 g||) = (||L_1 f||/||\ell_1||)$ and the first part is proved.

Write $M_1 f$ as $(W_2(u_1,f)/u_1) + (f/u_1)\, M_1 u_1$ where $u_1 = e^{\ell_1 t}$. These are the same operator since they both have leading coefficient one and they agree for $f = u_1$. This gives that

$$|M_1 f| \leq \left| \frac{W_2(u_1,f)}{u_1} \right| + \left| \frac{f}{u_1} M_1 u_1 \right|$$

and since $(W_2(u_1,f)/u_1) = L_1 f$ this is

$$\begin{aligned} |M_2 f| &\leq |L_1 f| + \left\| \frac{L_1 f}{\ell_1} \right\| \left| \frac{M_1 u_1}{u_1} \right| \\ &= |L_1 f| + \left\| \frac{L_1 f}{\ell_1} \right\| \; |\ell_1 + B(t)| \; . \end{aligned}$$

This completes the proof of this proposition.

Inequalities Involving a Second Order Operator

The next proposition concerns the operator L_2 and provides bounds for $|f|$, $|M_1 f|$ and $|M_2 f|$ if M_1 is any first order linear operator and M_2 is any second order linear operator.

PROPOSITION 3.6 *Let* $L_2f = (D-\ell_1)(D-\ell_2)f$ *where* $\ell_1 \leq \ell_2$ *are real numbers. Let* f *be twice differentiable on* $(-\infty, \infty)$ *and let* $M_1 = (D + B(t))$ *and* $M_2 = (D^2 + B_1(t)D + B_2(t))$ *then*

i) if $0 < \ell_1 < \ell_2$ *and* $\lim_{t \to \infty} e^{-\ell_1 t} f(t) = 0$

$$|f| \leq \left|\left| \frac{L_2f}{\ell_1\ell_2} \right|\right|$$

$$|M_1f| \leq \left|\left| \frac{L_2f}{\ell_1\ell_2} \right|\right| (\ell_1 + |\ell_1 + B(t)|)$$

and

$$|M_2f| \leq |L_2f| + \left|\left| \frac{L_2f}{\ell_1\ell_2} \right|\right| [\ell_1|\ell_1 + \ell_2 + B_1|$$

$$+ |\ell_1^2 + \ell_1B_1 + B_2|]$$

ii) if $\ell_1 < 0 < \ell_2$ *and* $\lim_{t \to -\infty} e^{-\ell_1 t} f(t) = 0$ *and* $\lim_{t \to \infty} e^{-\ell_2 t} f(t) = 0,$ *then*

$$|f| \leq \left|\left| \frac{L_2f}{\ell_1\ell_2} \right|\right|$$

$$|M_1f| \leq \left|\left| \frac{L_2f}{\ell_1\ell_2} \right|\right| \text{Min } (\ell_2 + |\ell_2 + B|, -\ell_1 + |\ell_1 + B|)$$

and

$$|M_2f| \leq |L_2f| + \left|\left| \frac{L_2f}{\ell_1\ell_2} \right|\right| [\ell_2|\ell_1 + \ell_2 + B_1| + |\ell_2^2 + \ell_2B_1 + B_2|]$$

also

$$|M_2f| \leq |L_2f| + \left|\left| \frac{L_2f}{\ell_1\ell_2} \right|\right| [|\ell_1| \, |\ell_1 + \ell_2 + B_1| + |\ell_1^2 + \ell_1B_1 + B_2|]$$

iii) if $0 < \ell_1 = \ell_2$ *and* $\lim_{t \to \infty} e^{-\ell_1 t} f(t) = 0$

$$|f| \leq \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\|$$

$$|M_1 f| \leq \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\| (\ell_1 + |\ell_1 + B|) \quad ,$$

and

$$|M_2 f| \leq |L_2 f| + \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\| [|\ell_1| \, |2\ell_1 + B_1| + |\ell_1^2 + \ell_1 B_1 + B_2|] \quad .$$

In all of these equality holds only if f *is constant.*

Proof. Let u_1, u_2 be a basis for the solution space of $L_2 f = 0$. Using u_1 and u_2 both M_1 and M_2 may be rewritten,

$$M_1 f = \frac{W_2(u_i, f)}{u_i} + \frac{f}{u_i} M_1 u_i \; ; \qquad i = 1,2$$

$$M_2 f = \frac{W_3(u_1, u_2, f)}{W_2(u_1, u_2)} + \frac{W_2(u_1, f)}{W_2(u_1, u_2)} [M_2 u_2 - \frac{u_2}{u_1} M_2 u_1]$$

where $u_i \neq 0$ and $W(u_1, u_2) \neq 0$. The proof of this proposition uses these expansions of M_1 and M_2 along with Corollary 3.2.

i) Since $\lim_{t \to \infty} e^{-\ell_1 t} f(t) = 0$, one has that $Z_f(\infty) \geq 2$ and if $g \equiv 1$, then $Z_g(\infty) \geq 2$ since $\lim_{t \to \infty} e^{-\ell_i t} = 0$, $i = 1,2,.$

Apply Corollary 3.2 twice with $t_i = \infty$ and $r_i = k$ taking the values first 1 and then 2. A basis for $\{G : L_2 G = 0; Z_G(\infty) \geq 1\}$ is $e^{\ell_1 t}$ and so Corollary 3.2 gives

$$|L_{2-1}f| \leq |L_{2-1}g| \left\| \frac{L_2 f}{L_2 g} \right\| \quad , \quad k = 1$$

$$|L_{2-2}f| \leq |L_{2-2}g| \left\| \frac{L_2 f}{L_2 g} \right\| \quad , \quad k = 2$$

which are

$$\left| \frac{W(u_1,f)}{u_1} \right| \leq \left| \frac{W(u_1,1)}{u_1} \right| \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\|$$

$$|f| \leq \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\|$$

respectively with $u_1 = e^{\ell_1 t}$. Letting $u_2 = e^{\ell_2 t}$ and using the triangle inequality on the expansions of M_1 and M_2 gives the inequalities of i).

ii) Corollary 3.2 applied here with $g = -1$ and $t_1 = -\infty$, $t_2 = \infty$, $r_1 = 1$ and $r_2 = 1$ gives that

$$|f| \leq |g| \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\| .$$

Applying the same corollary with $t_1 = -\infty$ and $r_1 = k = 1$ gives

$$\left| \frac{W_2(u_2,f)}{u_2} \right| \leq \left| \frac{W_2(u_2,g)}{u_2} \right| \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\|$$

and with $t_1 = \infty$, $r_1 = k = 1$ it yields

$$\left| \frac{W_2(u_1,f)}{u_1} \right| \leq \left| \frac{W_2(u_1,g)}{u_1} \right| \left\| \frac{L_2 f}{\ell_1 \ell_2} \right\|$$

with $u_1 = e^{\ell_1 t}$ and $u_2 = e^{\ell_2 t}$. Again using the expansion of M_1 and M_2 in terms of u_1 and u_2 and the triangle inequality gives the inequalities in ii).

iii) Corollary 3.2 applies here to every interval $[a, \infty]$, $a \neq -\infty$. This is a result of Corollary 3.4 and this restriction is necessary because, when $\ell_1 = \ell_2$, L_2 is no longer disconjugate on $[-\infty, \infty]$. At ∞, $e^{\ell_1 t}$, $te^{\ell_1 t}$ is a principal system of solutions for L_2 and so $\lim_{t \to \infty} e^{-\ell_1 t} f(t) = 0$ implies that $Z_f(\infty) \geq 2$. Once again $e^{\ell_1 t}$ is a basis for $\{G : L_2 G = 0,\ Z_G(\infty) \geq 1\}$ and so Corollary 3.2 with $g = 1$ gives

$$\left| \frac{W_2(u_1,f)}{u_1} \right| \leq \left| \frac{W_2(u_1,g)}{u_1} \right| \left|\left| \frac{L_2 f}{\ell_1 \ell_2} \right|\right|$$

with $k = 1$. For $k = 2$ the corollary again gives

$$|f| \leq |g| \left|\left| \frac{L_2 f}{\ell_1 \ell_2} \right|\right|$$

and as before these inequalities imply the inequalities of the proposition.

In i), ii), and iii) equality can hold only for $f = cg + c_1 G_1 + \ldots + c_{n-k} G_{n-k}$ from Corollary 3.2. However for equality to hold anywhere it must be that $|f| = ||(L_2 f)/(\ell_1 \ell_2)||$ and this happens only when $f = cg$, a constant.

These particular bounds and this method are found in [6] on page 118.

Inequalities Involving a Third Order Operator

The last proposition in this chapter deals with the third order operator $L_3 = (D-\ell_1)(D-\ell_2)(D-\ell_3)$ where ℓ_1, ℓ_2 and ℓ_3 are real numbers. In this case bounds on f, M_1f, M_2f, M_3f are given in terms of $||L_3f||$, ℓ_1, ℓ_2, ℓ_3, B_1, B_2 and B_3 where $M_1 = D + B_1(t)$, $M_2 = D^2 + B_1(t)D + B_2(t)$ and $M_3 = D^3 + B_1(t)D^2 + B_2(t)D + B_3(t)$.

PROPOSITION 3.7 *Let* $L_3 = (D-\ell_1)(D-\ell_2)(D-\ell_3)$ *with* $\ell_1 \leq \ell_2 \leq \ell_3$ *three real numbers and let* f *be three times differentiable on* $(-\infty,\infty)$

i) If $\lim_{t\to\infty} e^{-\ell_1 t} f(t) = 0$ *and* $0 < \ell_1 < \ell_2 < \ell_3$ *then*

$$|f| \leq \left|\left| \frac{L_3 f}{\ell_1\ell_2\ell_3} \right|\right|$$

$$|M_1 f| \leq \left|\left| \frac{L_3 f}{\ell_1\ell_2\ell_3} \right|\right| [\ell_1 + |\ell_1 + B_1(t)|]$$

$$|M_2 f| \leq \left|\left| \frac{L_3 f}{\ell_1\ell_2\ell_3} \right|\right| [\ell_1\ell_2 + \ell_1 | \ell_1 + \ell_2 + B_1| + |\ell_1^2 + \ell_1 B_1 + B_2|]$$

and

$$|M_3 f| \leq |L_3 f| + \left|\left| \frac{L_3 f}{\ell_1\ell_2\ell_3} \right|\right| [|\ell_1^3 + \ell_1^2 B_1 + \ell_1 B_2 + B_3|$$

$$+ |\ell_1((\ell_2^2 + \ell_1\ell_2 + \ell_1^2) + B_1(\ell_1 + \ell_2) + B_2)|$$

$$+ \left| \frac{\ell_1\ell_2(\ell_3^3(\ell_2-\ell_1) + \ell_1^3(\ell_3-\ell_2) - \ell_2^3(\ell_3-\ell_1))}{\ell_3^2(\ell_2-\ell_1) + \ell_1^2(\ell_3-\ell_2) - \ell_2^2(\ell_3-\ell_1)} + B_1\ell_1\ell_2 \right|]$$

ii) If $\lim_{t\to\infty} e^{-\ell_2 t} f(t) = \lim_{t\to-\infty} e^{-\ell_1 t} f(t) = 0$ *and*

$\ell_1 < 0 < \ell_2 < \ell_3$ *then*

$$|f| \leq \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\|$$

$$|M_1 f| \leq \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| [|\ell_i| + |\ell_i + B_1|] , \quad i = 1,2$$

$$|M_2 f| \leq \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| [|\ell_i \ell_j| + |\ell_i(\ell_i + \ell_j + B_1)| + |\ell_i^2 + \ell_i B_1(t) + B_2|]$$

$$\text{for} \quad i = 1, \quad j = 2 \quad \text{and} \quad i = 2, \; j = 3$$

and

$$|M_3 f| \leq |L_3 f| + \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| [|\ell_2^3 + \ell_2^2 B_1 + \ell_2 B_2 + B_3|$$

$$+ |\ell_2((\ell_3^2 + \ell_2 \ell_3 + \ell_3^2) + B_1(\ell_2 + \ell_3) + B_2|$$

$$+ \left| \frac{\ell_2 \ell_3 (\ell_1^3(\ell_3 - \ell_2) + \ell_2^3(\ell_1 - \ell_3) - \ell_3^2(\ell_1 - \ell_2))}{\ell_1^2(\ell_3 - \ell_2) + \ell_2^2(\ell_1 - \ell_3) - \ell_3^2(\ell_1 - \ell_3)} + B_1 \ell_2 \ell_3 \right|]$$

and the inequality on $|M_3 f|$ *from* i *is also true.*

iii) If $\lim e^{-\ell_1 t} f(t) = 0$ *and* $\ell_1 = \ell_2$ *with* $0 < \ell_1 < \ell_3$ *then*

$$|f| \leq \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\|$$

$$|M_1 f| \leq \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| [\ell_1 + |\ell_1 + B_1|]$$

$$|M_2 f| \leq \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| [\ell_1^2 + \ell_1 |2\ell_1 + B_1| + |\ell_1^2 + \ell_1 B_1 + B_2|]$$

and

$$|M_3 f| \le |L_3 f| + \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| \left[|\ell_1^3 + \ell_1^2 B_1 + \ell_1 B_2 + B_3| \right.$$

$$\left. + |\ell_1(3\ell_1^2 + 2\ell_1 B_1 + B_2)| + \left| \frac{\ell_1^2(\ell_3^2 - 3\ell_1^2\ell_3 + 2\ell_1^3)}{\ell_3^2 + \ell_1^2 - 2\ell_1\ell_3} + \ell_1^2 B_1 \right| \right.$$

iv) If $\lim_{t\to\infty} e^{-\ell_2 t} f(t) = \lim_{t\to-\infty} e^{-\ell_1 t} f(t) = 0$ *and* $\ell_1 < 0 < \ell_2 = \ell_3$ *then*

$$|M_2 f| \le \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| \left\| \left| \frac{(\ell_1^2 + B_1\ell_1 - B_2)\ell_2^2}{(\ell_2 - \ell_1)^2} \right| \right.$$

$$\left. + \left| \frac{\ell_1\ell_2(\ell_2^2 - 2\ell_1\ell_2 - \ell_1 B_1 - B_2)}{(\ell_2 - \ell_1)^2} \right| + \left| \frac{\ell_1(\ell_2^2 + \ell_2 B_1 + B_2)}{(\ell_2 - \ell_1)} \right| \right|$$

and

$$|M_3 f| \le |L_3 f|$$

$$+ \left\| \frac{L_3 f}{\ell_1\ell_2\ell_3} \right\| \left| \left| \frac{\ell_1\ell_2}{(\ell_2 - \ell_1)^2} \quad (2\ell_2^2(\ell_2 - \ell_1) + B_1(\ell_2^2 - 2\ell_2) - \ell_1 B_1 - B_3) \right| \right.$$

$$+ \left| \frac{\ell_2}{(\ell_2 - \ell_1)} (\ell_1^3 + \ell_1^2 B_1 + \ell_1 B_2 + B_3) \right|$$

$$\left. + \left| \frac{\ell_1}{(\ell_2 - \ell_1)} (\ell_2^3 + \ell_2^2 B_1 + \ell_2 B_2 + B_3) \right| \right|$$

v) If $\lim_{t\to\infty} e^{-\ell_1 t} f(t) = 0$ *and* $0 < \ell_1 = \ell_2 = \ell_3$ *then the equalities of iii) hold for* f, $M_1 f$ *and* $M_2 f$ *and*

$$|M_3 f| \le |L_3 f| + \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| [\ell_1^3 + \ell_1^2 B_1 + \ell_1 B_2 + B_3|$$

$$+ |\ell_1(3\ell_1^2 + 2\ell_1 B_1 + B_2)| + |\ell_1^2(6\ell_1 + B_1)|]$$

Proof. i) Let u_1, u_2, u_3 be a basis for the set of solutions of $L_3 x = 0$. Then whenever $u_1 \neq 0$ and $W(u_1,u_2) \neq 0$, $M_1 f$ and $M_2 f$ may be written as:

$$M_1 f = \frac{W_2(u_1,f)}{u_1} + \frac{f}{u_1} M_1 u_1$$

and

$$M_2 f = \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)} + \frac{W_2(u_1,f)}{W_2(u_1,u_2)} [M_2 u_2 - \frac{u_2}{u_1} M_2 u_1] + \frac{f}{u_1} M_2 u_1 .$$

Now due to the nature of the conditions imposed on f Corollary 3.2 can be used to obtain bounds on f, $W_2(u_1,f)$ and $W_3(u_1,u_2,f)$ when $u_1 = e^{\ell_1 t}$, $u_2 = e^{\ell_2 t}$ and $u_3 = e^{\ell_3 t}$. Since $\lim_{t \to \infty} e^{-\ell_1 t} f(t) = 0$ and since u_1, u_2, u_3 is a principal system at ∞, f has at least three zeros at ∞. Let $g = -1$ so that $L_3 g = \ell_1 \ell_2 \ell_3 > 0$ and $\lim_{t \to \infty} e^{-\ell_1 t} g = 0$. Then since L_3 is disconjugate on $[-\infty,\infty]$ and f and g have three zeros at ∞ the Corollary 3.2 gives that

$$|L_{3-k} f| \le |L_{3-k} g| \left\| \frac{L_3 f}{L_3 g} \right\|, \qquad k = 1,2,3.$$

The set $\{G : L_3 G = 0,\ Z_G(\infty) \geq 1\}$ has as a basis $\{e^{\ell_1 t}, e^{\ell_2 t}\}$. The set $\{G : L_3 G = 0,\ Z_G(\infty) \geq 2\}$ has a basis $\{e^{\ell_1 t}\}$ and therefore

$$L_{3-1}\, f = \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)}$$

$$L_{3-2}\, f = \frac{W_2(u_1,f)}{u_1}$$

and

$$L_{3-3}\, f = f \quad .$$

The inequalities from the corollary are this

$$\left| \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)} \right| \le \left| \frac{W_3(u_1,u_2,-1)}{W_2(u_1,u_2)} \right| \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\|$$

$$\left| \frac{W_2(u_1,f)}{u_1} \right| \le \left| \frac{W_2(u_1,-1)}{u_1} \right| \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\|$$

$$|f| \le \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| \, .$$

Combining the inequalities with the expansions of the operators M_1 and M_2 and using the triangle inequality gives

$$\begin{aligned} |M_1 f| &\le \left| \frac{W_2(u_1,f)}{u_1} \right| + |f| \left| \frac{M_1 u_1}{u_1} \right| \\ &= \left| \frac{W_2(u_1,f)}{u_1} \right| + |f| \, |\ell_1 + B_1(t)| \\ &\le \left[\left| \frac{W_2(u_1,-1)}{u_1} \right| + |\ell_1 + B_1(t)| \right] \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| \\ &= [\,|\ell_1| + |\ell_1 + B_1(t)|\,] \left\| \frac{L_3 f}{\ell_1 \ell_2 \ell_3} \right\| \end{aligned}$$

and

$$|M_2 f| \leq \left|\frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)}\right| + \left|\frac{W_2(u_1,f)}{W_2(u_1,u_2)}\right| \left|M_2u_2 - \frac{u_2}{u_1} M_2u_1\right| + |f| \left|\frac{M_2u_1}{u_1}\right|$$

$$\leq \left[\left|\frac{W_3(u_1,u_2,-1)}{W_2(u_1,u_2)}\right| + \left|\frac{W_2(u_1,-1)}{u_1}\right| |\ell_2 + \ell_1 + B_1| \right.$$

$$\left. + |\ell_1^2 + \ell_1 B_1 + B_2| \right] \left\| \frac{L_3 f}{\ell_1\ell_2\ell_3} \right\|$$

$$= [|\ell_1\ell_2^2 - \ell_1^2\ell_2| + |\ell_1|\ |\ell_1 + \ell_2 + B_1|$$

$$+ |\ell_1^2 + \ell_1B_1 + B_2|] \left\|\frac{L_3 f}{\ell_1\ell_2\ell_3}\right\| .$$

The inequality on $M_3 f$ is obtained by writing

$$M_3 f = \frac{W_4(u_1,u_2,u_3,f)}{W_3(u_1,u_2,u_3)} + \frac{W_3(u_1,u_2,f)}{W_3(u_1,u_2,u_3)} M_3u_3$$

$$+ \frac{W_3(u_1,f,u_3)}{W_3(u_1,u_2,u_3)} M_3u_2 + \frac{W_3(f,u_2,u_3)}{W_3(u_1,u_2,u_3)} M_3u_1 .$$

In order to use the bounds already found on $W_3(u_1,u_2,f)$, $W_2(u_1,f)$ and f it is necessary to write $W_3(u_1,f,u_3)$ and $W(f,u_2,u_3)$ in terms of $W_3(u_1,u_2,f)$, $W_2(u_1,f)$ and f. Using the method already employed in the case of $M_2 f$ gives

$$\frac{W_3(u_2,u_3,f)}{W_2(u_2,u_3)} = \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)} + \frac{W_2(u_1,f)}{W_2(u_1,u_2)} \left[- \frac{u_2}{u_1} \frac{W_3(u_2,u_3,u_1)}{W_2(u_2,u_3)}\right]$$

$$+ \frac{f}{u_1} \frac{W_3(u_2,u_3,u_1)}{W_2(u_2,u_3)}$$

and

$$\frac{W_3(u_1,u_3,f)}{W_2(u_1,u_3)} = \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)} + \frac{W_2(u_1,f)}{W_2(u_1,u_2)}\left[\frac{W_3(u_1,u_3,u_2)}{W_2(u_1,u_3)}\right] .$$

Combining all these gives

$$M_3 f = \frac{W_4(u_1,u_2,u_3,f)}{W_3(u_1,u_2,u_3)} + \frac{f}{u_1} M_3 u_1 + \frac{W_2(u_1,f)}{W_2(u_1,u_2)} \left[M_3 u_2 - \frac{u_2}{u_1} M_3 u_1\right]$$
$$+ \frac{W_3(u_1,u_2,f)}{W_3(u_1,u_2,u_3)W_2(u_1,u_2)} [M_3 u_3 W_2(u_1,u_2) - M_3 u_2 W_2(u_1,u_3)$$
$$+ M_3 u_1 W_2(u_2,u_3)]$$

and now applying the triangle inequality and the bounds on $W_3(u_1,u_2,f)$, $W_2(u_1,f)$ and f and recognizing $(W_4(u_1,u_2,u_3,f))/(W_3(u_1,u_2,u_3))$ as $L_3 f$ completes this first part.

ii) In this case L_3 is still disconjugate on $[-\infty, \infty]$ and $u_1 = e^{\ell_1 t}$, $u_2 = e^{\ell_2 t}$, $u_3 = e^{\ell_3 t}$ is a Descartes fundamental principal system of solutions for L_3 on $[-\infty, \infty]$. The conditions on f imply that f has at least two zeros at ∞ and one zero at $-\infty$.

To apply Corollary 3.2, let $g = 1$ so that $Z_g(\infty) \geq 2$ and $Z_g(-\infty) \geq 1$ while $L_3 g = -\ell_1\ell_2\ell_3 > 0$. Now applying that corollary with $t_1 = -\infty$, $r_1 = 1$ and $k = 1$ gives

$$|L_{3-1} f| \leq |L_{3-1} g| \left\| \frac{L_3 f}{L_3 g} \right\|$$

where

$$L_{3-1} f = \frac{W_3(u_2,u_3,f)}{W_2(u_2,u_3)}$$

since the set $\{G : L_3 G = 0;\ Z_G(-\infty) \geq 1\}$ has for a basis

$u_2 = e^{\ell_2 t}$, $u_3 = e^{\ell_3 t}$. Therefore

$$\left|\frac{W_3(u_2,u_3,f)}{W_2(u_2,u_3)}\right| \leq \left|\frac{W_3(u_2,u_3,1)}{W_2(u_2,u_3)}\right| \left\|\frac{L_3 f}{L_3 g}\right\|.$$

In the case $t_1 = \infty$, $r_1 = 1$, $k = 1$, Corollary 3.2 gives

$$|L_{3-1}\, f| \leq |L_{3-1}\, g| \left\|\frac{L_3 f}{L_3 g}\right\|$$

where $L_{3-1}\, f$ is now $(W_3(u_1,u_2,f))/(W_2(u_1,u_2))$.

With $t_1 = -\infty$, $t_2 = \infty$, $r_1 = 1$, $r_2 = 1$ and $k = 2$, Corollary 3.2 gives

$$|L_{3-2}\, f| \leq |L_{3-2}\, g| \left\|\frac{L_3 f}{L_3 g}\right\|$$

where, since $\{G : L_3 G = 0;\ Z_G(\infty) \geq 1,\ Z_G(-\infty) \geq 1\}$ has a basis $\{u_2 = e^{\ell_2 t}\}$, $L_{3-2}\, f = (W_2(u_2,f))/(u_2)$.

With $t_1 = \infty$, $k = 2$, $r_1 = 2$ one gets

$$|L_{3-2}\, f| \leq |L_{3-2}\, g| \left\|\frac{L_3 f}{L_3 g}\right\|$$

where $L_{3-2}\, f$ is now $(W_2(u_1,f))/(u_1)$.

Lastly if $k = 3$, $t_1 = -\infty$, $t_2 = \infty$, $r_1 = 1$, $r_2 = 2$, Corollary 3.2 gives

$$|f| \leq |g| \left\|\frac{L_3 f}{L_3 g}\right\|.$$

Because of this large number of inequalities there are more possible ways to rewrite M_1 and M_2. Letting $u_1 = e^{\ell_1 t}$, $u_2 = e^{\ell_2 t}$,

$u_3 = e^{\ell_3 t}$ one may write

$$M_1 f = \frac{W_2(u_i, f)}{u_i} + \frac{f}{u_i} M_1 u_i , \quad i = 1,2$$

and obtain inequalities on $|M_1 f|$ for $i = 1$ and 2.

Also

$$M_2 f = \frac{W_3(u_2,u_3,f)}{W_2(u_2,u_3)} + \frac{W_2(u_2,f)}{W_2(u_2,u_3)} [M_2 u_3 - \frac{u_3}{u_2} M_2 u_2] + \frac{f}{u_2} M_2 u_2$$

and the expression in i) for $M_2 f$ both give inequalities on $|M_2 f|$.

There is also a choice now in rewriting $M_3 f$. Of course the method of i) is available and works here to give the same inequality. However, the bounds on $W_3(u_2,u_3,f)$ and $W_2(u_2,f)$ allow $W_3(u_1,u_2,f)$ and $W_3(u_1,u_3,f)$ to be written in terms of $W_3(u_2,u_3,f)$ and $W_2(u_2,f)$ and f to get the same kind of inequality.

The inequality given however leaves $W_3(u_1,u_2,f)$ and $W_3(u_2,u_3,f)$ as they are in

$$M_3 f = \frac{W_4(u_1,u_2,u_3,f)}{W_3(u_1,u_2,u_3)} + \frac{W_3(u_1,u_2,f)}{W_3(u_1,u_2,u_3)} M_3 u_3$$

$$+ \frac{W_3(u_1,f,u_3)}{W_3(u_1,u_2,u_3)} M_3 u_2 + \frac{W_3(f,u_2,u_3)}{W_3(u_1,u_2,u_3)} M_3 u_1$$

and uses

$$\frac{W_3(u_1,u_3,f)}{W_2(u_1,u_3)} = \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)} + \frac{W_2(u_1,f)}{W_2(u_1,u_2)} \left[- \frac{W_3(u_1,u_2,u_3)}{W_2(u_1,u_3)} \right] .$$

This gives that

$$M_3 f = \frac{W_4(u_1,u_2,u_3,f)}{W_3(u_1,u_2,u_3)} + \frac{W_3(u_2,u_3,f)}{W_3(u_1,u_2,u_3)} M_3 u_1$$

$$+ \frac{W_3(u_1,u_2,f)}{W_3(u_1,u_2,u_3)} [M_3 u_3 - \frac{W_2(u_1,u_3)}{W_2(u_1,u_2)} M_3 u_2] + \frac{W_2(u_1,f)}{W_2(u_1,u_2)} M_3 u_2 \quad ,$$

and the triangle inequality and bounds on $W_3(u_2,u_3,f)$, $W_3(u_1,u_2,f)$, $W_2(u_1,f)$ are used as before to get the inequality.

iii) Here L_3 is no longer disconjugate on $[-\infty, \infty]$ and so Corollary 3.2 does not apply directly to $[-\infty, \infty]$. However Corollary 3.3 gives that L_3 is disconjugate on $[a, \infty]$ with $a \neq -\infty$. At ∞, $u_1 = e^{\ell_1 t}$, $u_2 = te^{\ell_1 t}$ and $u_3 = e^{\ell_3 t}$ is a principal system of solutions for L_3 and so $\lim_{t \to \infty} e^{-\ell_1 t} f(t) = 0$ implies that f has three zeros at ∞. Therefore apply Corollary 3.2 to L_3 on $[a, \infty]$ with $g = -1$. This gives

$$|L_{3-1} f| \leq |L_{3-1} g| \left\| \frac{L_3 f}{L_3 g} \right\|$$

where $L_{3-1} f = (W_3(u_1,u_2,f))/(W_2(u_1,u_2))$ when $k = 1$ and $t_1 = \infty$.

If $k = 2$, $t_1 = \infty$ and $r_1 = 2$ then Corollary 3.2 gives

$$|L_{3-2} f| \leq |L_{3-2} g| \left\| \frac{L_3 f}{L_3 g} \right\|$$

where $L_{3-2} f = (W_2(u_1,f))/(u_1)$.

Finally for $k = 3$, Corollary 3.2 gives

$$|L_{3-3} f| \leq |L_{3-3} g| \left\| \frac{L_3 f}{L_3 g} \right\|$$

which is just $|f| \leq \left|\left|\frac{L_3 f}{L_3 g}\right|\right|$.

These inequalities on $|f|$, $|W_2(u_1,f)|$ and $|W_3(u_1,u_2,f)|$ are now used with the expansions of $M_1 f$, $M_2 f$ and $M_3 f$ found in i). Any difference in form is a result of the fact that here $W_3(u_1,u_2,f) = W_3(e^{\ell_1 t}, te^{\ell_1 t} f)$ instead of $W_3(e^{\ell_1 t}, e^{\ell_2 t}, f)$.

iv) This case is essentially different from all the others because here there is no assumption that the function f has three zeros on any closed interval on which L_3 is disconjugate. By Corollary 3.3 L_3 is disconjugate on $[a, \infty]$ and $[-\infty, b]$ for $a \neq -\infty$ and $b \neq \infty$.

The system $e^{\ell_1 t}$, $e^{\ell_2 t}$, $te^{\ell_2 t}$ is a principal system for L_3 at ∞ and $-te^{\ell_2 t}$, $e^{\ell_2 t}$, $e^{\ell_1 t}$ is a principal system of solutions at $-\infty$. The condition $\lim_{t \to \infty} e^{-\ell_2 t} f(t) = 0$ means f has at least two zeros at ∞ and that $\lim_{t \to -\infty} e^{-\ell_1 t} f(t) = 0$ means f has at least one zero at $-\infty$. Let $g = 1$, then $L_3 g \geq 0$ and $Z_g(\infty) \geq 2$ while $Z_g(-\infty) \leq 1$. Applying Corollary 3.2 to $[a, \infty]$ gives

$$|L_{3-1} f| \leq |L_{3-1} g| \left|\left|\frac{L_3 f}{L_3 g}\right|\right|$$

where $k = 1$, $t_1 = \infty$ and $L_{3-1} f = (W_3(e^{\ell_1 t}, e^{\ell_2 t}, f))/(W_2(e^{\ell_1 t}, e^{\ell_2 t}))$ and

$$|L_{3-2} f| \leq |L_{3-2} g| \left|\left|\frac{L_3 f}{L_3 g}\right|\right|$$

where $k = 2$, $t_1 = \infty$ and $L_{3-2} f = (W_2(e^{\ell_1 t}, f))/(e^{\ell_1 t}))$.

Applying Corollary 3.2 to $[-\infty, b]$ gives

$$|L_{3-1} f| \leq |L_{3-1} g| \left\| \frac{L_3 f}{L_3 g} \right\|$$

when $k = 1$, $t_1 = -\infty$ and now

$$L_{3-1} f = \frac{W_3(e^{\ell_2 t}, te^{\ell_2 t}, f)}{W_2(e^{\ell_2 t}, te^{\ell_2 t})} .$$

There is however no inequality given on $|f|$ by this corollary and so the previous expansions of M_1, M_2 and M_3 do not work. However, $M_2 f$ may be written as

$$M_2 f = M_2 u_1 \frac{W_3(u_2,u_3,f)}{W_3(u_1,u_2,u_3)} + M_2 u_2 \frac{W_2(u_1,f)}{W_2(u_1,u_2)}$$

$$+ W_3(u_1,u_2,f) \left[\frac{M_2 u_3 \; W_2(u_1,u_2) - M_2 u_2 \; W_2(u_1,u_3)}{W_2(u_1,u_2) \; W_3(u_1,u_2,u_3)} \right]$$

for u_1, u_2, u_3 linearly independent. This is true because both sides agree on u_1, u_2 and u_3 and the coefficient of f'' on the right side is

$$\frac{M_2 u_1 \; W_2(u_2,u_3) + M_2 u_3 \; W_2(u_1,u_2) - M_2 u_2 \; W(u_1,u_3)}{W_3(u_1,u_2,u_3)} = 1 .$$

The terms u_i'' from $M_2 u_i$ gives $(W_3(u_1,u_2,u_3))/(W_3(u_1,u_2,u_3)) = 1$ and the remaining terms are

$$B_2(u_1' \; W_2(u_2.u_3) + u_3' \; W_2(u_1,u_2) - u_2' \; W_2(u_1,u_3)) = 0$$

$$B_1(u_1, \; W_2(u_2,u_3) + u_3 \; W_2(u_1,u_2) - u_2 \; W_2(u_1,u_3)) = 0 .$$

In this case $u_1 = e^{\ell_1 t}$, $u_2 = e^{\ell_2 t}$ and $u_3 = te^{\ell_2 t}$. Using this expansion and the bounds on $W_3(u_1,u_2,f)$, $W_3(u_2,u_3,f)$ and $W_2(u_1,f)$ gives the inequality.

The inequality on $M_3 f$ is had by rewriting $W_3(u_1,u_3,f)$ in

$$M_3 f = \frac{W_4(u_1,u_2,u_3,f)}{W_3(u_1,u_2,u_3)} + \frac{W_3(u_1,u_2,f)}{W_3(u_1,u_2,u_3)} M_3 u_3$$

$$+ \frac{W_3(u_1,f,u_3)}{W_3(u_1,u_2,u_3)} M_3 u_2 + \frac{W_3(f,u_2,u_3)}{W_3(u_1,u_2,u_3)}$$

using

$$\frac{W_3(u_1,u_3,f)}{W_2(u_1,u_3)} = \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)} + \frac{W_2(u_1,f)}{W_2(u_1,u_2)} \left[- \frac{W_3(u_1,u_2,u_3)}{W_2(u_1,u_3)} \right] .$$

v) This case is not essentially different from case iii) and the inequalities are arrived at in exactly the same way. The difference in the inequality on $|M_3 f|$ is due to the fact that in this case $u_1 = e^{\ell_1 t}$, $u_2 = te^{\ell_1 t}$ and $u_3 = t^2 e^{\ell_1 t}$.

This completes the proof of this proposition.

In the last proposition the function f is assumed to have three zeros. If f is assumed to have fewer than three zeros inequalities on certain special operators can be obtained.

For example if $M_2 f = f'' + B_1 f' + B_2 f$ and $0 < \ell_1 < \ell_2 < \ell_3$ and $L_3 f = (D-\ell_1)(D-\ell_2)(D-\ell_3)$ and $Z_f(\infty) = 2$ then Corollary 3.2 gives bounds on $W_2(u_1,f)$ and $W_3(u_1,u_2.f)$ though not on f itself. If M_2 is such that $M_2 u_1 = 0$ the expansion

$$M_2 f = \frac{W_3(u_1,u_2,f)}{W_2(u_1,u_2)} + \frac{W_2(u_1,f)}{W_2(u_1,u_2)} [M_2 u_2 - \frac{u_2}{u_1} M_2 u_1] + \frac{f}{u_1} M_2 u_1$$

can be used to get an inequality on $M_2 f$ in terms of $||(L_3 f)/(\ell_1 \ell_2 \ell_3)||$ ℓ_1, ℓ_2, ℓ_3 and $B_1(t)$ and $B_2(t)$.

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